Control chart analysis for a bulk arrival, non-Markovian queue, random breakdown, compulsory vacation, delay time, extended vacation and a stand by server

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Abstract

This article deals with a single server queue. Arrivals are in batches, follows compound Poisson process with each batch size is a random variable follows a discrete distribution. The customers are served singly, they wait in a queue of infinite capacity, if the service is not immediate. The server takes compulsory vacation after completing service, the server may breakdown while doing service. The service time, vacation time and repair time, all follows different general distributions. The number of breakdowns follows a Poisson distribution. After breakdown, the repair is carried out but repair is not immediately carried out, called a delay period it follows general distribution. Also, the completion of compulsory vacation, the vacation may be extended follows general distribution. The model is completely analysed using supplementary variable technique in steady state. Numerical illustrations are also provided. In addition, control chart analysis for number of customers in the system has been included.

Keywords: Non-Markovian queue - Bulk arrival - Compulsory vacation – Infinite capacity - FIFO - Extended vacation.

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1. Introduction

Many researchers consider vacation queues with variant vacation policies namely single vacation, Bernoulli's vacation, multiple vacation, compulsory vacation etc. A huge amount of vacation models is defined and analyzed by researchers. For a detailed survey one can refer Doshi (1986) and Takagi (1991). In a similar way many researchers working on bulk queue (either bulk arrival and /or bulk service). Some important reference in this area are Armero and Conesa (2000), Arumuganathan and Ramaswami (2005), Chang et al., (2004), Fakinos (1991), Srinivasan et al., (2002), Sumita and Masuda (1997), Ushakumari and Krishnamoorthy (1998), Stadje (1989), Ramaswami (1980), Lucantoni (1991) and many others.

The other topic in queueing theory, which has immediate practical application is queue with breakdown, also called queue with unreliable server. Many researchers have contributed on queue with unreliable servers and some noteworthy works are Ke (2005) and Wang (1995,1997).

When a system suddenly stops functioning due to a failure, most of the works available in the literature assume that the repair process on the system starts immediately, However Khalaf et al., (2012) analyzed a batch arrival breakdown and delay time. The concept of extended vacations and delay in starting the repair process, some authors introduced the idea of a stand-by server in some of the system (Madan, 1995)

2. The Model

A single server queue has been considered. In which the customers arrive in batches of random size X , with probability $Pr\{X = j\} = C_j$. The batch arrival follows Poisson process with parameter λ . The customers arrive for service, if service is not immediate, they will wait in a waiting line of an infinite capacity. The customers are served singly. The service time follows general distribution with distribution function $B(x)$. The services are given First Come First Service basis. At each service completion the server takes vacation of random length, follows general distribution with distribution function $G_1(x)$. The system may breakdown at random and the number of breakdowns follows Poisson process with parameter α . Once the system breakdown, the customer whose service is interrupted goes to the head of the queue. After breakdown, there is a potential delay before the repair period starts. The delay period follows general distribution with distribution function $D(x)$. After the delay time the server is send for repair, the repair period follows general distribution with distribution function $R(x)$. After a vacation period, the server has the option of taking an extended vacation with probability r . The extended vacation period follows general distribution with distribution function $G_2(x)$. When the server is on vacation or repair, there is a standby server, the server serves with distribution function negative exponential of rate μ_2. The mathematical model is defined using the notations given in the following table 2.1.

Table 2.1: Notations

Notations	Description
λ	Arrival rate
X	Arrival batch service
C_j	$Pr\{X = j\}$
$B(x)$	Service time distribution
$G_1(x)$	Vacation time distribution
α	Breakdown rate
$D(x)$	Delay time distribution
$G_2(x)$	Extended vacation time distribution
$R(x)$	Repair time distribution
γ	Probability of taking extended vacation
μ_2	Stand by server's service rate
$\xi(t)$	The server state at time t
$\mu_1(x)$	Hazard rate function of service time
$\theta_1(x)$	Hazard rate function of vacation time
$\theta_2(x)$	Hazard rate function of extended vacation time
$\gamma(x)$	Hazard rate function of repair time
$\beta(x)$	Hazard rate function of delay time

Since $G_1(x), G_2(x), B(x), D(x)$ and $R(x)$ are cumulative distribution functions, we have

$$G_1(0) = G_2(0) = B(0) = D(0) = R(0) = 0$$
$$G_1(\infty) = G_2(\infty) = B(\infty) = D(\infty) = R(\infty) = 0$$

The Hazard rate function of service time, vacation time, extended vacation time, repair time, delay time are defined as

$$\mu_1(x) = \frac{dB(x)}{1 - B(x)}, \theta_1(x) = \frac{dG_1(x)}{1 - G_1(x)}, \quad \theta_2(x) = \frac{dG_2(x)}{1 - G_2(x)}, \gamma(x) = \frac{dR(x)}{1 - R(x)}, \beta(x) = \frac{dD(x)}{1 - D(x)}$$

We define the following generating functions,

$$P(z;x)=\sum_{n=0}^{\infty}P_n(x)z^n\ ,V(z;x)=\sum_{n=0}^{\infty}V_n(x)z^n\,,R(z;x)=\sum_{n=0}^{\infty}R_n(x)z^n$$

$$S(z;x)=\sum_{n=0}^{\infty}S_n(x)z^n,U(z;x)=\sum_{n=0}^{\infty}U_n(x)z^n$$

Define $M(t)$ be the number of customers in the queue at time t and the server state $\xi(t)$, the server state at time t.

$$\xi(t)=\begin{cases}0, & \text{server is idle at time } t\\ 1, & \text{server is busy at time } t\\ 2, & \text{server is on vacation at time } t\\ 3, & \text{server is on extended vacation at time } t\\ 4, & \text{server is in delay at time } t\\ 5, & \text{server is in repair at time } t\end{cases}$$

$\{M(t),\xi(t):t\geq 0\}$ is a bivariate Markov process, where $c(t)=0,b_1(t),b_2(t),b_3(t),b_4(t)$ $,b_5(t)$

The following probabilities are defined for the mathematical analysis

$$Q(t)=Pr\{M(t)=0;c(t)=0\}$$
$$P_n(x;t)=Pr\{M(t)=n;t\leq b_1(x)\leq t+dt\}$$
$$V_n(x;t)=Pr\{M(t)=n;t\leq b_2(x)\leq t+dt\}$$
$$S_n(x;t)=Pr\{M(t)=n;t\leq b_3(x)\leq t+dt\}$$
$$U_n(x;t)=Pr\{M(t)=n;t\leq b_4(x)\leq t+dt\}$$
$$R_n(x;t)=Pr\{M(t)=n;t\leq b_5(x)\leq t+dt\}$$

The steady state probabilities are as $t\to\infty$

$P_n(x;t)\to P_n(x);\ V_n(x;t)\to V_n(x);\ S_n(x;t)\to S_n(x);\ U_n(x;t)\to U_n(x);$ $R_n(x;t)\to R_n(x);$

Based on the arguments in Cox (1955), the following differential-difference equations are obtained

$$\frac{\partial}{\partial x}P_0(x)=-(\lambda+\mu_1(x)+\alpha)P_0(x) \tag{2.1}$$

$$\frac{\partial}{\partial x}P_n(x)=-(\lambda+\mu_1(x)+\alpha)P_n(x)+\lambda\sum_{i=1}^{n}C_iP_{n-i}(x),\ \text{for } n\geq 1 \tag{2.2}$$

$$\frac{\partial}{\partial x}V_0(x)=-(\lambda+\theta_1(x)+\mu_2)V_0(x)+\mu_2\ V_1(x) \tag{2.3}$$

$$\frac{\partial}{\partial x}V_n(x)=-(\lambda+\theta_1(x)+\mu_2)V_n(x)+\lambda\sum_{i=1}^{n}C_iV_{n-i}(x)+\mu_2V_{n+1}\ ,\ \text{for } n\geq 1 \tag{2.4}$$

$$\frac{\partial}{\partial x}R_0(x)=-(\lambda+\gamma(x)+\mu_2)R_0(x)+\mu_2\ R_1(x) \tag{2.5}$$

$$\frac{\partial}{\partial x}R_n(x)=-(\lambda+\gamma(x)+\mu_2)R_n(x)+\lambda\sum_{i=1}^{n}C_iR_{n-i}(x)+\mu_2R_{n+1},\ \text{for } n\geq 1 \tag{2.6}$$

$$\frac{\partial}{\partial x}S_0(x)=-(\lambda+\theta_2(x)+\alpha)S_0(x) \tag{2.7}$$

$$\frac{\partial}{\partial x}S_n(x)=-(\lambda+\theta_2(x)+\alpha)S_n(x)+\lambda\sum_{i=1}^{n}C_iS_{n-i}(x),\ \text{for } n\geq 1 \tag{2.8}$$

$$\frac{\partial}{\partial x}U_0(x)=0 \tag{2.9}$$

$$\frac{\partial}{\partial x}U_n(x) = -\big(\lambda+\beta(x)\big)U_n(x) + \lambda\sum_{i=1}^{n} C_i U_{n-i}(x), \text{ for } n \geq 1 \tag{2.10}$$

$$\lambda Q = \int_0^\infty R_0(x)\gamma(x)dx + (1-p)\int_0^\infty P_0(x)\mu_1(x) + (1-r)\int_0^\infty V_0(x)\theta_1(x)dx + \int_0^\theta S_0(x)\theta_2(x)dx \tag{2.11}$$

The Boundary conditions are

$$P_n(0) = (1-p)\int_0^\infty P_{n+1}(x)\mu_1(x)dx + (1-r)\int_0^\infty V_{n+1}(x)\theta_1(x)dx + \int_0^\infty S_{n+1}(x)\theta_2(x)dx + \int_0^\infty R_{n+1}(x)\gamma(x)dx + \lambda C_{n+1}Q, \text{ for } n \geq 0 \tag{2.12}$$

$$V_n(0) = p\int_0^\infty P_n(x)\mu_1(x)dx, \text{ for } n \geq 0 \tag{2.13}$$

$$S_n(0) = r\int_0^\infty V_n(x)\theta_1(x)dx, \text{ for } n \geq 0 \tag{2.14}$$

$$U_n(0) = \int_0^\infty P_{n-1}(x)\alpha dx = \alpha P_{n-1}(x), \text{ for } n \geq 1 \tag{2.15}$$

$$R_n(0) = \int_0^\infty U_n(x)\beta(x)dx, \text{ for } n \geq 1 \tag{2.16}$$

$$R_n(0) = U_n(0) = 0 \tag{2.17}$$

Multiplying equation (2.2) by z^n and applying $\sum_{n=1}^{\infty}$, we have

$$\frac{\partial}{\partial x}\sum_{n=1}^{\infty} P_n(x)z^n = -(\lambda+\mu_1(x)+\alpha)\sum_{n=1}^{\infty} P_n(x)z^n + \lambda\sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i P_{n-i}(x)z^n$$

Adding the above equation with equation (2.1), we have

$$\frac{\partial}{\partial x}\sum_{n=0}^{\infty} P_n(x)z^n = -(\lambda+\mu_1(x)+\alpha)\sum_{n=0}^{\infty} P_n(x)z^n + \lambda\sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i P_{n-i}(x)z^n$$

Integrating of above equation with respect to x with the limits from $'0'$ to $'x'$, we have

$$P(z;x) = P(z;0)e^{[-(\lambda+\alpha)+\lambda C(z)]x-\int_0^x \mu_1(u)du} \tag{2.18}$$

Multiplying equation (2.4) by z^n and applying $\sum_{n=1}^{\infty}$, we have

$$\frac{\partial}{\partial x}\sum_{n=1}^{\infty} V_n(x)z^n = -(\lambda+\theta_1(x)+\mu_2)\sum_{n=1}^{\infty} V_n(x)z^n + \lambda\sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i V_{n-i}(x)z^n + \mu_2\sum_{n=1}^{\infty} V_{n+1}(x)z^n$$

Adding the above equation with equation (2.3), we have

$$\frac{\partial}{\partial x}\sum_{n=0}^{\infty} V_n(x)z^n = -(\lambda+\theta_1(x)+\mu_2)\sum_{n=0}^{\infty} V_n(x)z^n + \lambda\sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i V_{n-i}(x)z^n + \mu_2\sum_{n=1}^{\infty} V_{n+1}(x)z^n$$

Integrating of above equation with respect to x with the limits from $'0'$ to $'x'$, we have

$$V(z;x) = V(z;0)e^{\left[-(\lambda+\mu_2)+\lambda C(z)+\frac{\mu_2}{z}\right]x-\int_0^x \theta_1(u)du} \tag{2.19}$$

Multiplying equation (2.6) by z^n and applying $\sum_{n=1}^{\infty}$, we have

$$\frac{\partial}{\partial x}\sum_{n=1}^{\infty} R_n(x)z^n = -(\lambda+\gamma(x)+\mu_2)\sum_{n=1}^{\infty} R_n(x)z^n + \lambda\sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i R_{n-i}(x)z^n + \mu_2\sum_{n=1}^{\infty} R_{n+1}(x)z^n$$

Adding the above equation with equation (2.5), we have

$$\frac{\partial}{\partial x}\sum_{n=0}^{\infty} R_n(x)z^n = -(\lambda+\gamma(x)+\mu_2)\sum_{n=0}^{\infty} R_n(x)z^n + \lambda\sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i R_{n-i}(x)z^n + \mu_2\sum_{n=1}^{\infty} R_{n+1}(x)z^n$$

Integrating of above equation with respect to x with the limits from $'0'$ to $'x'$, we have

$$R(z;x) = R(z;0)e^{\left[-(\lambda+\mu_2)+\lambda C(z)+\frac{\mu_2}{z}\right]x-\int_0^x \gamma(u)du} \tag{2.20}$$

Multiplying equation (2.8) by z^n and applying $\sum_{n=1}^{\infty}$, we have

$$\frac{\partial}{\partial x}\sum_{n=1}^{\infty} S_n(x)z^n = -\left(\lambda+\theta_2(x)\right)\sum_{n=1}^{\infty} S_n(x)z^n + \lambda\sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i S_{n-i}(x)z^n$$

Adding the above equation with equation (2.7), we have

$$\frac{\partial}{\partial x}\sum_{n=0}^{\infty} S_n(x)z^n = -\left(\lambda+\theta_2(x)\right)\sum_{n=0}^{\infty} S_n(x)z^n + \lambda\sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i S_{n-i}(x)z^n$$

Integrating of above equation with respect to x with the limits from $'0'$ to $'x'$, we have

$$S(z;x) = S(z;0)e^{[-(\lambda+\lambda C(z))]x-\int_0^x \theta_2(u)du} \tag{2.21}$$

Multiplying equation (2.10) by z^n and applying $\sum_{n=1}^{\infty}$, we have

$$\frac{\partial}{\partial x}\sum_{n=1}^{\infty} U_n(x)z^n = -\left(\lambda+\beta(x)\right)\sum_{n=1}^{\infty} U_n(x)z^n + \lambda\sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i U_{n-i}(x)z^n$$

Adding the above equation with equation (2.9), we have

$$\frac{\partial}{\partial x}\sum_{n=0}^{\infty} U_n(x)z^n = -\left(\lambda+\beta(x)\right)\sum_{n=0}^{\infty} U_n(x)z^n + \lambda\sum_{n=1}^{\infty}\sum_{i=1}^{n} C_i U_{n-i}(x)z^n$$

Integrating of above equation with respect to x with the limits from $'0'$ to $'x'$, we have

$$U(z;x) = U(z;0)e^{[-(\lambda+\lambda C(z)]x-\int_0^x \beta(u)du} \tag{2.22}$$

Multiplying equation (2.12) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$P(z;0) = \frac{1-p}{z}\int_0^\infty P(z;x)\mu_1(x)dx + \frac{1-r}{z}\int_0^\infty V(z;x)\theta_1(x)dx + \frac{1}{z}\int_0^\infty S(z;x)\theta_2(x)dx + \frac{1}{z}\int_0^\infty R(z;x)\gamma(x)dx + \frac{\lambda}{z}c(z)Q - \frac{1-p}{z}\int_0^\infty P_0(x)\mu_1(x)dx - \frac{1-r}{z}\int_0^\infty V_0(x)\theta_1(x)dx - \frac{1}{z}\int_0^\infty S_0(x)\theta_2(x)dx - \frac{1}{z}\int_0^\infty R_0(x)\gamma(x)dx \quad (2.23)$$

Substituting the value of equation (2.11) in (2.23), we have

$$P(z;0) = \frac{1-p}{z}\int_0^\infty P(z;x)\mu_1(x)dx + \frac{1-r}{z}\int_0^\infty V(z;x)\theta_1(x)dx + \frac{1}{z}\int_0^\infty \theta_2(x) S(z;x)dx + \frac{1}{z}\int_0^\infty R(z;x)\gamma(x)dx + \frac{\lambda Q}{z}(c(z)-1) \quad (2.24)$$

Multiplying equation (2.13) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$V(z;0) = p\int_0^\infty \mu_1(x)P(z;x)dx \quad (2.25)$$

Multiplying equation (2.14) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$S(z;0) = r\int_0^\infty \theta_1(x)v(z;x)dx \quad (2.26)$$

Multiplying equation (2.15) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$U(z;0) = \alpha(z)P(z) \quad (2.27)$$

Multiplying equation (2.16) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$R(z;0) = \int_0^\infty \gamma(x)U(z;x)dx \quad (2.28)$$

Integrating equations (2.18) partially with respect to ‘x’, with the limits from ‘0’ to ‘x’, we have

$$P(z;x)dx = P(z;0)e^{Lx-\int_0^x \mu_1(u)du} \quad (2.29)$$

where, $L = [-(\lambda+\alpha) + \lambda C(z)]$

Multiplying equation (2.25) by $\mu_1(x)$ and integrating partially with respect to ‘x’, with the limits from ‘0’to ‘∞’, we have

$$\int_0^\infty P(z;x)\mu_1(x)dx = P(z;0)B^*(L) \quad (2.30)$$

Integrating equation (2.25) partially with respect to ‘x’, with the limits from ‘0’ to ‘∞’, we have

$$P(z) = \int_0^\infty P(z;x)dx = P(z;0)\frac{[1-B^*(L)]}{L} \quad (2.31)$$

Integrating equation (2.19) partially with respect to ‘x’, with the limits from ‘0’ to ‘x’, we have

$$V(z;x)dx = V(z;0)e^{Ax-\int_0^x \theta_1(u)du} \quad (2.32)$$

where, $A = [-(\lambda+\mu_2) + \lambda C(z) + \frac{\mu_2}{z}]$

Multiplying equation (2.28) by $\theta_1(x)$ and integrating partially with respect to ‘x’, with the limits from ‘0’to ‘∞’, we have

$$\int_0^\infty V(z;x)\theta_1(x)dx = V(z;0)G_1^*(A) \tag{2.33}$$

Integrating equation (2.28) partially with respect to ‘x’, with the limits from ‘0’ to ‘∞’, we have

$$V(z) = \int_0^\infty V(z;x)dx = V(z;0)\frac{[1 - G_1^*(A)]}{A} \tag{2.34}$$

Integrating equation (2.20) partially with respect to ‘x’, with the limits from ‘0’ to ‘x’, we have

$$S(z;x)dx = S(z;0)e^{Hx-\int_0^x \theta_2(u)du} \tag{2.35}$$

where, $H = [-\lambda + \lambda C(z)]$

Multiplying equation (2.31) by $\theta_2(x)$ and integrating partially with respect to ‘x’, with the limits from ‘0’to ‘∞’, we have

$$\int_0^\infty S(z;x)\theta_2(x)dx = S(z;0)G_2^*(H) \tag{2.36}$$

Integrating equation (2.31) partially with respect to ‘x’, with the limits from ‘0’ to ‘∞’, we have

$$S(z) = \int_0^\infty S(z;x)dx = S(z;0)\frac{[1-G_2^*(H)]}{H} \tag{2.37}$$

Integrating equation (2.21) partially with respect to ‘x’, with the limits from ‘0’ to ‘x’, we have

$$R(z;x)dx = R(z;0)e^{Ax-\int_0^x \gamma(u)du} \tag{2.38}$$

Multiplying equation (2.34) by $\gamma(x)$ and integrating partially with respect to ‘x’, with the limits from ‘0’to ‘∞’, we have

$$\int_0^\infty R(z;x)\gamma(x)dx = R(z;0)R^*(A) \tag{2.39}$$

Integrating equation (2.34) partially with respect to ‘x’, with the limits from ‘0’ to ‘∞’, we have

$$R(z) = \int_0^\infty R(z;x)dx = V(z;0)\frac{[1-R^*(A)]}{A} \tag{2.40}$$

Integrating equation (2.22) partially with respect to ‘x’, with the limits from ‘0’ to ‘x’, we have

$$U(z;x)dx = U(z;0)e^{Hx-\int_0^x \beta(u)du} \tag{2.41}$$

Multiplying equation (2.37) by $\beta(x)$ and integrating partially with respect to ‘x’, with the limits from ‘0’to ‘∞’, we have

$$\int_0^\infty U(z;x)\beta(x)dx = U(z;0)D^*(H) \tag{2.42}$$

Integrating equation (2.37) partially with respect to ‘x’, with the limits from ‘0’ to ‘∞’, we have

$$\int_0^\infty U(z;x)dx = U(z;0)\frac{[1-D^*(H)]}{H} \tag{2.43}$$

Substituting the values of equations (2.30), (2.33), (2.36), (2.39) in (2.24), we have

$$P(z;0) = \frac{1-p}{z}P(z;0)B^*(L) + \frac{1-r}{z}V(z;0)G_1^*(A) + \frac{1}{z}S(z;0)G_2^*(H) + \frac{1}{z}R(z;0)R^*(A) + \frac{\lambda Q}{z}(c(z)-1) \qquad (2.44)$$

Substituting the value of equation (2.30) in (2.13), we have

$$V(z;0) = pP(z;0)B^*(L) \qquad (2.45)$$

Substituting the value of equation (2.33) in (2.14), we have

$$S(z;0) = rV(z;0)G_1^*(A) \qquad (2.46)$$

Substituting the value of equation (2.45) in (2.46), we have

$$S(z;0) = rpP(z;0)B^*(L)G_1^*(A) \qquad (2.47)$$

Substituting the value of equation (2.31) in (2.15), we have

$$U(z;0) = \alpha z P(z;0)\frac{[1-B^*(L)]}{L} \qquad (2.48)$$

Substituting the value of equation (2.42) in (2.16), we have

$$R(z;0) = U(z;0)D^*(H) \qquad (2.49)$$

Substituting the value of equation (2.48) in (2.49), we have

$$R(z;0) = \alpha z P(z;0)\frac{[1-B^*(L)]}{L}D^*(H) \qquad (2.50)$$

Substituting the value of equation (2.45) , (2.47), (2.50) in (2.44), we have

$$P(z;0) = \frac{L\lambda Q(c(z)-1)}{J} \qquad (2.51)$$

where, $J = [zL - L(1-p)B^*(L) - L(1-r)pB^*(L)G_1^*(A) - LrpB^*(L) G_1^*(A)G_2^*(H) - z[1-B^*(L)]D^*(H)R^*(A)$

Now we have to find $H(z)$ by adding equations (2.31), (2.34), (2.37), (2.40), (2.42), we have

$$H(z) = P(z) + V(z) + R(z) + S(z) + U(z)$$

$$H(z) = P(z;0)\frac{K}{LAH} \qquad (2.52)$$

where, $K = AH[1-B^*(L)] + LH\,pB^*(L)[1-G_1^*(A)][1-B^*(L)]\alpha HD^*(H)[1-R^*(A)] + rpLAB^*(L)G_1^*(A)[1-G_2^*(H)] + \alpha ZA[1-B^*(L)][1-D^*(H)]$

Substituting the value of equation (2.51) in (2.52), we have

$$H(z) = \frac{\lambda Q(C(z)-1)K}{AHJ} \qquad (2.53)$$

where, $n_1 = \lambda Q(C(z)-1); n_2 = K;\ d_1 = A; d_2 = H; d_3 = J$

3. Some performance measures

The following performance measures are derived for the model discussed in section 2

1. Idle probability

$$Q = \frac{i_1}{i_2} \tag{3.1}$$

where, $\boldsymbol{i_1} = 12q^2\{d_1'd_2'd_3'\}^2$

$$\boldsymbol{i_2} = 12q^2\{d_1'd_2'd_3'\}^2 + 3\lambda(2 - q - \lambda q)q^2 n_2'''d_1'd_2'd_3' + 2\lambda q n_2''d_1'd_2'd_3' - 3\lambda q n_2''d_1''d_2'd_3' - 3\lambda q n_2''d_1'd_2''d_3' - 3\lambda q n_2''d_1'd_2'd_3''$$

2. Mean number of customers in the queue

$$M = H'(z) = H'(1) = \frac{D'''N'^{V} - N'''D'^{V}}{4(D''')^3} \tag{3.2}$$

$$M = \frac{6d_1'd_2'd_3'(6n_1''n_2'' + 4n_1'n_2''') - 3n_1'n_2''(12d_1''d_2'd_3' + d_1'd_2''d_3' + d_1'd_2'd_3'')}{4(6d_1'd_2'd_3')^2}$$

where, $\boldsymbol{d_1'} = \lambda E[X] + \mu_2$; $\boldsymbol{d_2'} = \lambda E[X]$; $\boldsymbol{d_1''} = \boldsymbol{d_2''} = \lambda(E[X]^2 - \lambda E[X])$; $\boldsymbol{n_1'} = Q\boldsymbol{d_2'}$;

$\boldsymbol{d_3'} = d_2'e_1 + \alpha p B^*(\alpha)E[V][d_1'] + \alpha r p B^*(\alpha)E[S]d_2' + \alpha E[U]d_2'e_1 + \alpha E[R]d_1'e_1 + \alpha B^*(\alpha)$ $\boldsymbol{n_1''} = Q\boldsymbol{d_2''}$,

$\boldsymbol{d_3''} = d_2' + d_1''B^*(\alpha) + d_2'B^*(\alpha) + d_2'B^*(\alpha) - 2d_2'B'^*(\alpha) + d_2' + 2d_2' pB^*(\alpha)E[V]d_1' + 2\alpha pB'^*(\alpha)d_2'E[V]d_1' - \alpha pB^*(\alpha)E[V^2][d_1']^2 + \alpha pB^*(\alpha)E[V]\lambda E[X^2] - \alpha pB^*(\alpha)E[V]d_2' + 2d_2'rpB^*(\alpha)E[S] d_2' + 2\alpha rpB'^*(\alpha)[d_2']^2E[S] - \alpha rpB^*(\alpha)E[V]E[S]d_2'd_1' - \alpha rp (\lambda E[X]^2)B^*(\alpha)E[S^2] + \alpha rpB^*(\alpha)E[S]\lambda E[X^2] - \alpha rpB'^*(\alpha)E[S] d_2' + 3\alpha E[U]d_2' + 2\alpha E[R]d_1' - \alpha E[U^2]\lambda E[X]^2 + \alpha E[U]\lambda E[X^2] 2\alpha E[U]d_2'E[R] - d_1' - \alpha E[R^2][\lambda E[X] + \mu_2]^2 + \alpha E[R]\lambda E[X^2] E[R] - \alpha E[R]d_2' + 2\alpha B'^*(\alpha)d_2' - 2\alpha B^*(\alpha)E[U]d_2' - 2\alpha B^*(\alpha) E[R]d_1' - 2\alpha B'^*(\alpha)\lambda E[X]^2 + \alpha B^*(\alpha)E[U^2]\lambda E[X]^2 - \alpha B^*(\alpha) E[U]\lambda E[X^2] + \alpha B^*(\alpha)E[U]d_2' + 2\alpha B^*(\alpha)E[U]d_2'E[R]d_1' + \alpha B^*(\alpha)E[R^2][d_1']^2 - \alpha B^*(\alpha)E[R]\lambda E[X]^2 + \alpha B^*(\alpha)E[R]d_2'$

$\boldsymbol{n_2''} = 2d_1'd_2'e_1 + 2\alpha d_1'e_1E[R]d_1' + 2rp\alpha d_1'B^*(\alpha)E[S]d_2' + 2\alpha d_1'e_1E[U]d_2'$;

$\boldsymbol{n_2'''} = 3d_1''d_2'e_1 + 3d_1'd_1''e_1 - 6d_1'[d_1']^2B'^*(\alpha) + 6[d_1']^2pB^*(\alpha)E[V]d_1' + 3\alpha d_1''pB^*(\alpha)E[V]d_1' + 6\alpha[d_1']^2pB'^*(\alpha)E[V]d_1' - 3\alpha d_2'p [d_1']^2 B^*(\alpha)E[V^2] + 6\alpha d_2'e_1E[R]d_1' + 3\alpha d_1''e_1E[R]d_1' - 6\alpha[d_2']^2 d_1' E[R]B'^*(\alpha) - 6\alpha[d_1']^2e_1E[U]E[R]d_1' - 3\alpha d_2'e_1 E[R^2] [d_1']^2 + 3 \alpha d_2'e_1E[R] d_1'' + 6rp[d_2']^2d_1'E[S] B^*(\alpha) + 3rp\alpha d_1''E[S] B^*(\alpha)d_2' + 6\alpha rp[d_2']^2d_1'E[S]B'^*(\alpha) - 6\alpha rp[d_1']^2E[S]E[V]B^*(\alpha) d_2' - 3\alpha rpd_1'B^*(\alpha)E[S^2][d_2']^2 + 3\alpha rpd_1'B^*(\alpha)d_1''E[S] + 6d_1'e_1E[U] d_2' + 2\alpha d_1''e_1E[U]d_2' - 6\alpha d_1'B'^*(\alpha)E[U][d_2']^2 - 3\alpha d_1'e_1E[U^2][d_2']^2 + 3\alpha d_1' e_1E[U]d_1''$

3. Variance number of customers in the queue

$$V = H''(1) + H'(1) - [H'(1)]^2 \tag{3.3}$$

$$H''(z) = H''(1) = \frac{h_1}{840(D''')^3} \tag{3.4}$$

where, $\boldsymbol{h_1} = 74[D''']^2N^V - 105[D'''D'^VN'^V] - 84[D'''D^VN'''] + 10[D''']^2N'^V + 105[D'^V]^2N'''$

$\boldsymbol{d_3'''} = 3d_1'' + e_2 - e_2B^*(\alpha) - 3d_1''B'^*(\alpha)d_2' - 3d_2'^3B''^*(\alpha) - 3d_2'd_1'' B'^*(\alpha) + 3d_1'' pB^*(\alpha)E[V]d_1' + 6[d_2']^2pB'^*(\alpha)E[V]d_1' - 3d_2'p B^*(\alpha)E[V^2][d_1']^2 + 3d_2'pB^*(\alpha) E[V]d_1'' + 3\alpha pB''^*(\alpha)[d_2']^2$

$$E[V]d_1' + 3\alpha pB'^*(\alpha)d_1''E[V]d_1' - 3\alpha pB'^*(\alpha)d_2'E[V^2][d_1']^2$$
$$+3\alpha pB'^*(\alpha)d_2'E[V]d_1'' + \alpha pB^*(\alpha)E[V^3][d_1']^3 - 3\alpha pB^*(\alpha)$$
$$E[V^2]d_1'd_1'' + \alpha pB^*(\alpha)E[V]e_2 3d_1''rpB^*(\alpha)E[S]d_2' + 6[d_2']^3$$
$$pB'^*(\alpha)E[S] - 4[d_2']^2 rpB^*(\alpha)E[V]d_1'E[S] - 3[d_2']^3 rpB^*(\alpha)$$
$$E[S] + 3d_2'\, rpB^*(\alpha)E[S]d_1'' + 3\alpha rpB''^*(\alpha)[d_2']^3 E[S] + 3\alpha r$$
$$pB'^*(\alpha)d_1''E[S]d_2' - 3\alpha rpB'^*(\alpha)[d_2']^3\, E[V]E[S]d_1' - 3\alpha rp$$
$$B'^*(\alpha)[d_2']^3 E[S^2] + 3\alpha rpB'^*(\alpha)\, E[S]d_2'd_1'' + \alpha rpB^*(\alpha)E[V^2]$$
$$[d_1']^2 E[S]d_2' - \alpha rpB^*(\alpha)E[V]d_1''E[S]d_2' + 2\alpha rpB^*(\alpha)E[V][d_1']^2$$
$$E[S^2][d_2']^2 - 2\alpha rpB^*(\alpha)E[V]d_1'E[S]d_1'' + \alpha rpB^*(\alpha)E[S^3][d_2']^3$$
$$-3\alpha rpB^*(\alpha)E[S^2]d_2'd_1'' + \alpha rpB^*(\alpha)E[S]e_2 - 3\alpha E[U^2][d_2']^2$$
$$+3\alpha E[U]d_1'' - 6\alpha E[U]d_2'E[R]d_1' - 3\alpha[d_1']^2 E[R^2] + 3\alpha E[R]d_1''$$
$$+\alpha E[U^3][d_2']^3 - 3\alpha E[U^2]d_2'd_1'' + 3\alpha E[U^2][d_2']^2 E[R]d_1' + \alpha$$
$$E[U]e_2 - 3\alpha E[U]d_1''E[R]d_1' + 3\alpha E[U]d_2'E[R^2][d_1']^2 - 3\alpha E[U]$$
$$d_2'E[R]d_1''[d_1']^3 + \alpha E[R^3] - 3\alpha E[R^2]d_1'd_1'' + \alpha E[R]e_2 + \alpha[d_2']^2$$
$$B''^*(\alpha) + 3\alpha B'^*(\alpha)d_1'' - 6\alpha B'^*(\alpha)[d_2']^2 E[U] - 6\alpha B'^*(\alpha)d_2'd_1'$$
$$E[R] + 3\alpha B^*(\alpha)E[U^2][d_2']^2 - 3\alpha B^*(\alpha)E[U]d_1'' + 6\alpha B^*(\alpha)$$
$$E[U]E[R]d_1' + 3\alpha B^*(\alpha)E[R^2][d_1']^2 - 3\alpha B^*(\alpha)E[R]d_1'' - 3\alpha$$
$$B''^*(\alpha)[d_2']^2 E[U]d_2' - 3\alpha B''^*(\alpha)[d_2']^2 E[R]d_1' - 3\alpha B'^*(\alpha)d_1''$$
$$E[U]d_2' - 3\alpha B'^*(\alpha)d_1''d_1'E[R] - 3\alpha B'^*(\alpha)E[U^2][d_2']^3 - 3\alpha$$
$$B'^*(\alpha)E[U]d_2'd_1'' + 6\alpha B'^*(\alpha)[d_2']^2 E[U]E[R]d_1' + 3\alpha B'^*(\alpha)$$
$$E[R^2]d_2'd_1' - 3\alpha B'^*(\alpha)E[R]d_2'd_1'' - \alpha B^*(\alpha)E[U^3][d_2']^3 - \alpha$$
$$B^*(\alpha)E[U^2]d_2'd_1'' - \alpha B^*(\alpha)E[U][d_2']^3 - 3\alpha B^*(\alpha)E[U]E[R]$$
$$d_1'd_1'' - 2\alpha B^*(\alpha)E[U^2][d_2']^2 E[R]d_1' - 3\alpha B^*(\alpha)\, E[U]E[R^2]d_2'$$
$$[d_1']^2 + 3\alpha B^*(\alpha)E[U]E[R]d_2'd_1'' - \alpha B^*(\alpha)\, E[R^3][d_1']^3 + 3\alpha$$
$$B^*(\alpha)E[R^2]d_1'd_1'' - \alpha B^*(\alpha)E[R]\, e_2$$

4. Numerical Illustrations:

The model analysed in this article is numerically analysed by assuming the general distribution has negative exponential distribution with following suitable parameters. Using the formulas: $E[X] = \frac{1}{q}, E[V] = \frac{1}{\theta_1} E[S] = \frac{1}{\theta_2} E[R] = \frac{1}{\gamma} E[U] = \frac{1}{\beta}, E[X^2] = \frac{2-q}{q^2}$,

$$E[V^2] = \frac{2-\theta_1}{{\theta_1}^2}, E[S^2] = \frac{2-\theta_2}{{\theta_2}^2}, E[R^2] = \frac{2-\gamma}{\gamma^2}, E[R^2] = \frac{2-\beta}{\beta^2},$$
$$B^*(\alpha) = \frac{\mu_1}{\alpha+\mu_1}, B'^*(\alpha) = \frac{-\mu_1}{(\alpha+\mu_1)^2}, B''^*(\alpha) = \frac{2\mu_1}{(\alpha+\mu_1)^3}$$

In section 3, the following performance measures are calculated

1. Idle probability

$$Q = \frac{i_1}{i_2}$$

where, $\boldsymbol{i_1} = 12q^2\{d_1'd_2'd_3'\}^2$

$$\boldsymbol{i_2} = 12q^2\{d_1'd_2'd_3'\}^2 + 3\lambda(2-q-\lambda q)q^2 n_2'''d_1'd_2'd_3' + 2\lambda q n_2''d_1'\,d_2'\,d_3' - 3\lambda q n_2''d_1''d_2'd_3' - 3\lambda q n_2''d_1'd_2''d_3' - 3\lambda q n_2''d_1'd_2'd_3''$$

2. Mean number of customers in the queue

$$M = \frac{6{d_1}'{d_2}'{d_3}'(6{n_1}''{n_2}'' + 4{n_1}'{n_2}''') - 3{n_1}'{n_2}''(12{d_1}''{d_2}'{d_3}' + {d_1}'{d_2}''{d_3}' + {d_1}'{d_2}'{d_3}'')}{4(6{d_1}'{d_2}'{d_3}')^2}$$

where, $\boldsymbol{d_1'} = \frac{g_2}{q}, g_2 = (\lambda + q\mu_2); \boldsymbol{d_2'} = \frac{\lambda}{q}; \boldsymbol{d_1''} = \boldsymbol{d_2''} = \lambda \frac{[2-q-\lambda q]}{q^2}, \boldsymbol{d_3'} = \frac{A_1}{q g_1 j_1},$

$\boldsymbol{j_1} = \theta_1\theta_2\beta\gamma;\ \boldsymbol{g_1} = \alpha + \mu_1;\ \boldsymbol{g_2} = (\lambda + q\mu_2);\ \boldsymbol{g_3} = (2-q);\ \boldsymbol{g_4} = (2-\gamma);$
$\boldsymbol{g_5} = (2-\theta_1);\ \boldsymbol{g_6} = (2-\theta_2);\ \boldsymbol{g_7} = (2-\beta);\ \boldsymbol{j_2} = {\theta_1}^2{\theta_2}^2\beta^2\gamma^2;\ \boldsymbol{j_3} = \theta_1{\theta_2}^2\beta^2\gamma^2;$
$\boldsymbol{j_4} = {\theta_1}^2\theta_2\beta^2\gamma^2;\ \boldsymbol{j_5} = {\theta_1}^2{\theta_2}^2\beta\gamma^2;\ \boldsymbol{j_6} = {\theta_1}^2{\theta_2}^2\beta^2\gamma;\ \boldsymbol{d_3''} = \frac{A_2 - A_3}{q^2 g_1^2 j_2}$

$\boldsymbol{A_1} = \alpha\lambda j_1 + \alpha p\mu_1 g_2\theta_2\beta\gamma + \alpha\lambda r p\mu_1\theta_1\beta\gamma + \alpha^2\lambda\theta_1\theta_2\gamma + \alpha^2 g_2\theta_1\theta_2\beta + \alpha\mu_1 j_1$

$\boldsymbol{A_2} = \lambda q g_1^2 j_2 + \lambda g_3 g_1^2 j_2 + \lambda\mu_1 q g_1 j_2 + 2\lambda^2 j_2 + 2\lambda p\mu_1 g_2 g_1\, j_3 + \alpha\lambda p\mu_1 g_3 g_1\, j_3$
$+2\lambda^2 r p\mu_1 g_1\, j_4 + \alpha\lambda r p\mu_1 g_3 g_1\, j_4 + 3\alpha\lambda q g_1^2 j_5 + 2\alpha g_2 g_1^2 j_6 + \alpha\lambda g_3 g_1^2 j_5 +$
$\alpha\lambda g_3 g_1^2 j_6 + 2\alpha\mu_1\lambda^2 j_5 + 2\alpha\mu_1\lambda g_2 j_6 + \alpha\mu_1\lambda^2 g_7 g_1 {\theta_1}^2{\theta_2}^2\gamma^2 + \alpha\mu_1\lambda q g_1 j_5 +$
$2\alpha\mu_1\lambda g_2 g_1 {\theta_1}^2{\theta_2}^2\beta\gamma + \alpha\mu_1 g_4 g_2^2 g_1 {\theta_1}^2{\theta_2}^2\beta^2 + 2\mu_1\lambda q g_1 j_6$

$\boldsymbol{A_3} = \mu_1 g_3 g_1\, j_2 + 2\alpha p\mu_1\lambda g_2\, j_3 + \alpha p\mu_1 g_5 g_2^2 g_1 {\theta_2}^2\beta^2\gamma^2 + \alpha p\mu_1\lambda q g_1\, j_3 + 2\alpha$
$r p\mu_1\lambda^2 j_4 + \lambda\alpha r p\mu_1 g_2 g_1\theta_1\theta_2\beta^2\gamma^2 + \alpha r p\mu_1\lambda^2 g_6 g_1 {\theta_1}^2\beta^2\gamma^2 + \lambda\alpha r p\mu_1 g_1\, j_4$
$+\alpha\lambda^2 g_7 g_1^2 {\theta_1}^2{\theta_2}^2\gamma^2 + 2\alpha\lambda g_2 g_1^2 {\theta_1}^2{\theta_2}^2\beta\gamma + \alpha g_4 g_1^2 g_2^2 {\theta_1}^2{\theta_2}^2\beta^2 + \alpha\lambda g_1^2 j_6$
$+2\alpha q\mu_1\lambda j_2 + 2\alpha q\mu_1\lambda g_1\, j_5 + 2\alpha q\mu_1 g_2 g_1\, j_6 + \alpha\mu_1\lambda g_3 g_1\, j_5 + \alpha\mu_1\lambda g_3 g_1\, j_6$

$\boldsymbol{n_2''} = \frac{B_1}{q^2 g_1\, j_1}$

$\boldsymbol{B_1} = 2\alpha\lambda g_2\, j_1 + 2\alpha\lambda p\mu_1 g_2\theta_2\beta\gamma + 2\alpha^2\lambda\theta_1\theta_2\beta g_2 + 2rp\alpha\mu_1\lambda\theta_1\beta\gamma g_2 +$
$2\alpha^2\lambda\theta_1\theta_2\gamma g_2$

$\boldsymbol{n_2'''} = \frac{B_2 - B_3}{q^3 {g_1}^2\, j_1}$; $\boldsymbol{r_1} = [2 - q - \lambda q]$

$\boldsymbol{B_2} = 3\alpha\lambda^2 r_1 g_1\, j_1 + 3\alpha\lambda g_2 r_1 g_1\, j_1 + 6\lambda^2\mu_1 g_2\, j_1 + 6\lambda^2 p q\mu_1 g_2 g_1 j_3 + 3\alpha p\mu_1\lambda$
$g_2 g_1 r_1 j_3 + 3\alpha\lambda^2 p\mu_1 r_1 g_1 j_3 + 6\alpha^2\lambda q g_2 g_1 j_6 + 3\alpha^2\lambda g_2 r_1 g_1\, j_6 + 6\alpha\lambda^2\mu_1$
$g_2 j_6 + 3\alpha^2\lambda^2 r_1 g_1 j_6 + 6rp\lambda^2\mu_1 g_2 g_1\, j_4 + 3rp\alpha\mu_1\lambda^2 r_1 g_1 j_4 + 3\alpha r p\mu_1\lambda$
$g_2 r_1 g_1\, j_4 + 6\alpha\lambda g_2 q g_1\, j_5 + 2\alpha^2\lambda^2 r_1 g_1 j_4 + 6\alpha\mu_1\lambda^2 g_2\, j_5 + 3\alpha^2 g_2 r_1 g_1 j_5$

$\boldsymbol{B_3} = 6\alpha\lambda^2 p\mu_1 g_2\, j_2 + 3\alpha\lambda p q\mu_1 g_5 g_2^2 {\theta_2}^2\beta^2\gamma^2 + 6\alpha^2\lambda^2 g_2 g_1 {\theta_1}^2{\theta_2}^2\beta\gamma + 3\alpha^2\lambda g_4$
$g_2^2 g_1 {\theta_1}^2{\theta_2}^2\beta^2 + 6\alpha r p\mu_1\lambda^2 g_2\, j_4 + 6\alpha r p\lambda\mu_1 g_1 g_2^2\theta_1\theta_2\beta^2\gamma^2 + 3\alpha r p\mu_1\lambda^2 g_2$
$g_6 g_1 {\theta_1}^2\beta^2\gamma^2 + 3\alpha^2\lambda^2 g_2 g_1 g_7 {\theta_1}^2{\theta_2}^2\gamma^2$

3. Variance number of customers in the queue

$$V = H''(1) + H'(1) - [H'(1)]^2$$

$$H''(z) = H''(1) = \frac{h_1}{840(D''')^3}$$

where, $\boldsymbol{d_3'''} = \frac{C_1 - C_2}{q^3 g_1^3 k_1}$; $\boldsymbol{r_2} = [6 - 6q + q^2]$; $\boldsymbol{r_3} = [6 - 6\theta_1 + {\theta_1}^2]$;
$\boldsymbol{r_4} = [6 - 6\theta_2 + {\theta_2}^2]; \boldsymbol{r_5} = [6 - 6\beta + \beta^2]; \boldsymbol{r_6} = [6 - 6\gamma + \gamma^2];$
$\boldsymbol{k_2} = {\theta_1}^2{\theta_2}^3\beta^3\gamma^3; \boldsymbol{k_3} = {\theta_1}^3{\theta_2}^2\beta^3\gamma^3; \boldsymbol{k_4} = {\theta_1}^3{\theta_2}^3\beta^2\gamma^3; \boldsymbol{k_5} = {\theta_1}^3{\theta_2}^3\beta^3\gamma^2$

$\boldsymbol{C_1} = 3\lambda q r_1 k_1 g_1^3 + \lambda r_2 k_1 g_1^3 + 3\lambda\mu_1 q g_3 k_1 g_1^2 + 6\lambda\mu_1 k_1 r_1 g_1 + 3p\mu_1 k_2 r_1 g_2 g_1^2 +$
$3\lambda p q\mu_1 k_2 r_1 g_1^2 + 6\alpha q p\mu_1\lambda^2 k_2 g_2 + 3\alpha p\lambda\mu_1 g_5 g_1 g_2^2\theta_1{\theta_2}^3\beta^3\gamma^3 + 6\alpha p\mu_1 r_3$
$g_1 g_2^3 {\theta_2}^3\beta^3\gamma^3 + \alpha p\lambda\mu_1 r_2 k_1 g_1^3 + 3\lambda r p\mu_1 k_3 r_1 g_1 + 6\alpha r p\mu_1\lambda^3 k_3 + 3\alpha r p\mu_1$
$\lambda^3 g_6 g_1 {\theta_1}^3\theta_2\beta^3\gamma^3 + \alpha r p\lambda\mu_1 q g_5 g_1^2 g_2^2 {\theta_1}^2\theta_2\beta^3\gamma^3 + 2\alpha r p q\mu_1\lambda^2 g_2 g_6 g_1^2\theta_2$
${\theta_1}^2\beta^3\gamma^3 + \alpha r p\mu_1\lambda^3 r_4 g_1^2 {\theta_1}^3\beta^3\gamma^3 + \alpha r p\mu_1\lambda r_2 k_1 g_1^3 + 6\alpha q r_1 k_1 g_1^3 + \alpha\lambda^3 g_1^3$
${\theta_1}^3{\theta_2}^3\gamma^3 r_5 + 3\alpha\lambda g_7 r_1 {\theta_1}^3{\theta_2}^3\gamma^3 g_1^3 + 3\alpha q\lambda^2 g_7 g_2 g_1^3 {\theta_1}^3{\theta_2}^3\beta\gamma^2 + \alpha\lambda r_2 k_4$
$g_1^3 + 3\alpha\lambda g_4 g_1^3 g_2^2 {\theta_1}^3{\theta_2}^3\beta^2\gamma + \alpha r_6 g_2^3 q^2 {\theta_1}^3{\theta_2}^3\beta g_1^3 + \alpha\lambda r_2 k_4 g_1^3 + 2\alpha\mu_1\lambda^2$

$$qk_1 + 6\alpha\mu_1\lambda^2 qg_1k_5 + 6\alpha\lambda\mu_1 qg_2k_5g_1 + 3\alpha\mu_1\lambda^2 g_4q^2{\theta_1}^3{\theta_2}^3\gamma g_1^2 + 6\alpha q \mu_1\lambda g_2g_1^2{\theta_1}^3{\theta_2}^3\beta^2\gamma^2 + 3\alpha\mu_1 g_4g_2^2 q{\theta_1}^3{\theta_2}^3\beta^3\gamma g_1^2 + 3\alpha\mu_1\lambda r_1g_1k_4 + 3\alpha\mu_1 g_2r_1k_5g_1 + 3\alpha\mu_1\lambda qr_1k_4g_1 + 3\alpha\mu_1\lambda r_1qk_5g_1 + 3\alpha\mu_1\lambda g_7r_1q{\theta_1}^3{\theta_2}^3\beta\gamma^3g_1^2 + 3\alpha\mu_1\lambda qg_3g_1^2k_4 + 2\alpha\mu_1\lambda q^2g_1^2k_4 + 3\alpha\mu_1\lambda g_1^2k_4r_1 + 3\alpha\mu_1 g_4g_2r_1{\theta_1}^3{\theta_2}^3 \beta^3\gamma g_1^2 + 3\lambda q\alpha\mu_1 g_3k_5g_1^2 + 2\lambda q^2\alpha\mu_1 k_5g_1^2$$

$$\boldsymbol{C_2} = 3\lambda qg_3k_1g_1^3 + 2\lambda q^2g_1^3 + \mu_1\lambda k_1r_2g_1^2 + 2\lambda q^2k_1g_1^2 + 6\lambda^3\mu_1q^2k_1 + 6\lambda^2p\mu_1 g_2g_1k_4 + 3\lambda p\mu_1g_5g_1^2g_2^2q\theta_1{\theta_2}^3\beta^3\gamma^3 + 3\alpha p\mu_1qg_2r_1g_1k_2 + 3\alpha pq\mu_1\lambda r_1k_2g_1^2 + 3\alpha pq\mu_1\lambda g_5g_2r_1g_1^2\theta_1{\theta_2}^3\beta^3\gamma^3 + 3\lambda qk_1g_3g_1^3 + 2\lambda q^2g_1^3k_1 + 3r pq\mu_1\lambda r_1k_1 g_1^2 + 6\lambda^3rp\mu_1k_3g_1 + 4\lambda^2rp\mu_1g_2g_1^2{\theta_1}^2{\theta_2}^2\beta^3\gamma^3 + 3\lambda^3 rp \mu_1g_6g_1^2{\theta_1}^3\theta_2\beta^3\gamma^3 + 3\alpha rpq\mu_1\lambda k_3r_1g_1 + 3\alpha rp\mu_1\lambda k_3r_1g_1 + \alpha rp\mu_1\lambda r_1 g_1^2{\theta_1}^2{\theta_2}^2\beta^3\gamma^3 + 2\alpha rp\mu_1g_2r_1qg_1^2{\theta_1}^2{\theta_2}^2\beta^3\gamma^3 + 3\alpha rp\mu_1\lambda g_6r_1g_1^2{\theta_1}^3\theta_2 \beta^3\gamma^3 + 3\alpha rpq\mu_1\lambda g_3k_1g_1^3 + 2\alpha rp\mu_1\lambda q^2k_1g_1^3 + 6\alpha\lambda qg_2g_1^3{\theta_1}^3{\theta_2}^3\beta^2\gamma^2 + 3\alpha g_4g_2^2q^2g_1^3{\theta_1}^3{\theta_2}^3\beta^3\gamma + 3q\alpha\lambda g_3k_4g_1^3 + 2\lambda q^2\alpha k_4 + 3\alpha g_2g_1^3{\theta_1}^3{\theta_2}^3 \beta^2\gamma^2r_1 + 3\alpha\lambda qg_1^3{\theta_1}^3{\theta_2}^3\beta^2\gamma^2r_1 + 3\alpha\, g_4g_2r_1g_1^3{\theta_1}^3{\theta_2}^3\beta^3\gamma + 3\lambda q g_3k_5g_1^3 + 2\lambda\alpha q^2g_1^3{\theta_1}^3{\theta_2}^3\beta^3\gamma + 3\alpha\mu_1qr_1g_1k_1\ 6\alpha\mu_1qr_1k_5g_1^2 + 6\alpha\mu_1 \lambda^3k_1 + 6\alpha\mu_1\lambda g_2k_5 + 3\alpha\mu_1\lambda^3g_7g_1{\theta_1}^3{\theta_2}^3\beta\gamma^3 + 6\alpha\mu_1\lambda^2g_2g_1{\theta_1}^3{\theta_2}^3\beta^2 \gamma^2 + 3\alpha\mu_1\lambda qg_4g_1g_2{\theta_1}^3{\theta_2}^3\beta^3\gamma + \alpha\mu_1\lambda^3r_5g_1^2{\theta_1}^3{\theta_2}^3\gamma + \alpha\mu_1\lambda r_2k_4g_1^2 + 3\alpha\mu_1g_1g_2r_1{\theta_1}^3{\theta_2}^3\beta^2\gamma^2 + 2\alpha\mu_1\lambda^2g_2g_7{\theta_1}^3{\theta_2}^3\beta\gamma^2g_1^2 + 3\alpha\mu_1\lambda g_4 g_1^2g_2^2{\theta_1}^3{\theta_2}^3\beta^2\gamma + \alpha\mu_1r_6g_1^2g_2^3{\theta_1}^3{\theta_2}^3\beta^3 + \alpha\mu_1\lambda r_2k_5g_1^2$$

The calculated values are tabulated in the table 4.1 and 4.2.

Table 4.1: Idle Probabilities **Table 4.2: Performance measures**

($\alpha = 3.99, \beta = 3.5, \gamma = 3.9, \theta_1 = 0.8, \theta_2 = 0.7, q = 0.7, p = 0.9, \mu_1 = 4.79, \mu_2 = 4.98, r = 0.9$)

λ	Q
1.1	0.06467
1.2	0.05987
1.3	0.05570
1.4	0.05203
1.5	0.04879
1.6	0.04590
1.7	0.04330
1.8	0.04095
1.9	0.03882
2.0	0.03687

λ	M	V
1.1	1.01586	82.9070
1.2	1.01524	85.3896
1.3	1.01481	87.9581
1.4	1.01452	90.6041
1.5	1.01433	93.3212
1.6	1.01420	96.1037
1.7	1.01411	98.9476
1.8	1.01405	101.8492
1.9	1.01401	104.8060
2.0	1.01398	107.8156

In table 4.1, for various arrival rates, the idle probabilities are calculated. As the arrival rate increases, the frequency of server becomes idle also increases. In the table 4.2, the performance measures, the mean number of customer and the variance number of customers are calculated and presented. As the arrival rate increases, the mean values decrease steadily. The variance increases considerably.

5. Control chart analysis

Statistical process control is very much useful in studying quality control of a system. Many methods are proposed and analyzed by researchers; one is using control chart analysis. A few common characteristics of control chart analysis, whatever may be the type, contains upper and lower control limits. The quality of the data is measured using the limit values. In this analysis there is a control line, which is usually considered to be the target value. Statistically controlled system, the observations lie nearer to the control limit (CL) and within the Upper control limit (UCL) and Lower control limit (LCL). The upper control limit and the lower control limit are measured by the following formulas:

$$CL = M;\ UCL = M + 3\sqrt{V};\ LCL = M - 3\sqrt{V}$$

For our model the control limits are calculated for mean number of customers in the system by assuming the batch size follows decapitated geometric distribution

$$a_k = \alpha(1-\alpha)^{k-1}, k = 0,1,2 \ldots .0 < \alpha < 1$$

The first moment is $\frac{1}{\alpha}$ and the second moment is $\frac{2(1-\alpha)}{\alpha^2}$

The calculated values are tabulated in the table 5.1

Table 5.1: Control limits

($\alpha = 3.99, \beta = 3.5, \gamma = 3.9, \theta_1 = 0.8, \theta_2 = 0.7, q = 0.7, p = 0.9, \mu_1 = 4.79, \mu_2 = 4.98, r = 0.9$)

λ	CL	UCL	LCL
1.1	1.01586	28.3318	0
1.2	1.01524	28.7372	0
1.3	1.01481	29.1506	0
1.4	1.01452	29.5703	0
1.5	1.01433	29.9952	0
1.6	1.01420	30.4239	0
1.7	1.01411	30.8558	0
1.8	1.01405	31.2901	0
1.9	1.01401	31.7264	0
2.0	1.01398	32.1642	0

The table 5.1, shows the control limit for various values of arrival rate. For lower limit, the calculated values are negative, in terms the values are taken as zero.

6. Conclusion

In this paper a single server batch arrival queue has been considered. In addition, the server takes compulsory vacation during service, the server may breakdown. Except inter arrival time and inter breakdown period random variable all other random variables in this model are generally distributed. That is the inter arrival time and inter breakdown period are negative exponential distributions. The model is analyzed in steady state by applying probability generation function method. In addition, statistical quality control process is carried out by the way of control chart analysis for mean number of customers in the system. Some numerical illustrated are obtained. The model can be extended by assuming general inter arrival time and/are generally distributed inter breakdown period.

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