# A Compulsory Vacation Unreliable Server, Bulk queue with Two types of Services/Repairs, State dependent arrival rates

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**Abstract**

A single unreliable server queue has been considered in this article. The arrival process follows compound Poisson process with state dependent arrival rates. Services are given in batches of size K of two types; each type is generally distributed. After completion of first type service (essential service), the batch enter into the second type service (optional service) based on a Bernoulli distribution. And the end of each service completion, the server takes a vacation of random period, generally distributed. While, if the service is going on, the server may breakdown and the number of breakdowns follows a Poisson distribution. Immediately the server undergoes two types of repairs, each repair period follows general distribution. After the completion of first repair (essential repair), the server undergoes the second repair (optional repair) based on a Bernoulli process. The system contains a queue of infinite size. This model is completely analyzed by introducing supplementary variables and using probability generating function technique. Some particular model and some system performance measures are derived. To show the applicable stability of the model some numerical illustrations are also provided.

**Keywords:** Non-Markovian queue - Bulk arrival queue - Compulsory vacation - Essential and Optional service/reapair - Unreliable server - State dependent rates.
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## 1. Introduction

A queueing system is a set of interconnected components like service seekers, working place and service providers, and working together to achieve a specific goal or purpose. The service seekers may be human customers or messages to be answered or programs to be processed, etc., The working place may be a working station or computer system etc., The service providers may be human servers or machines etc., In real life situations, if the providers are machines, it may frequently out of order, in general called breakdown. In such a case the machine may be replaced or repaired. But system manager's point of view, the first preference is repair. With this in mind many researchers working on queue with breakdown. Some notable works are Gaver (1962), Avi-Itzhak and Naor (1963), Thirurengadan (1963), Mitrany and Avi-Itzhak (1968), Sengupta (1990), Li et al., (1997), Tang (1997) and Takine and Sengupta (1997).

Vacation queueing models with different arrival pattern are studied by many authors, including Baba (1986), Choudhury (2000), Choudhury and Borthskur (2000), Lee and Srinivasan (1989), Lee et al (1995), Rosenberg and Yechiali (1993) and Teghem (1990) and many others have studied batch arrival vacation queues under different vacation policies. Harris (1967) analyzed a queueing model, in which the arrival and service parameter depends on current state. In 2008, Kerner considered a non-Markovian queue with state dependent parameters.

In the practical situations like, hospitals, productions systems the clients some time need additional service other than the regular service. Some notable works in recent years are Medhi (2002), Wang (2004), Choudhury and Deka (2012). Chen and Renshaw (2004) analyzed a

Markovian bulk queue together with a control policy at idle time. Chen et al., (2010) analyzed a Markovian queue with state dependent control policy.

This paper contains five sections. In the second section, the model definition and analysis are given. In the third section, some system statistical constants are given. In the fourth section, some numerical illustrations are provided. In the last section a conclusion is given.

## 2. The Mathematical definitions and notations

In this section, the model has been defined and relevant notations are introduced. A single server non-Markovian queue has been considered with the following characteristics:

The client arrives are in groups of variable size $j$ , $j = 1, 2, 3, \ldots$ with probability distribution $C_j$. The services are given in batches of fixed size K. There are two types of services, called essential service (ES) and optional service (OS). After completing ES, the batch OS with probability $q$ $(0 \leq q \leq 1)$ or leaves the system with may demand probability $(1-q)$. The service periods $S_1$(ES) and $S_2$ (OS) are generally distributed with distribution functions $G_1(x)$ and $G_2(x)$ respectively. The total service time of a batch is $S = (1-q)S_1 + qS_2$. There is a queue with infinite capacity. After completion of each service, the server takes vacation of compulsory type, the random vacation period $V$ follows general distribution with distribution function is $B(x)$. While server is busy, the server may breakdown, the breakdown period follows negative exponential with mean $\frac{1}{\alpha}$. Immediately, the server undergoes repair process, the essential repair period (ER) follows a general distribution $H_1(x)$. In addition, after completion of the repair period, the server undergoes another repair process called optional repair (OR) with probability $r$ or the server enter into the system with probability $(1-r)$ $(0 \leq r \leq 1)$. The optional repair period follows general distribution function with distribution function $H_2(x)$. The arrival rate is $\lambda = \lambda_i$; i = 0, during idle period; i = 1, during ES period; i = 2, during OS period; i = 3, during vacation period; i = 4, during ER period; i = 5, during OR period. The mean batch size, the mean total service period, the mean vacation period and mean repair period are respectively E(X), E(S), E(V ) and $E(R)$.

$\mu_j(x) = \frac{g_j(x)}{1-G_j(x)}; j = 1(ES), 2(OS)$ that the conditional probability completion of service period during the interval $(x, x+dx)$ given that the elapsed service time of the batch in service is $x$ .

$\beta(x) = \frac{b(x)}{1-\boldsymbol{B}(x)}$; that the conditional probability completion of vacation period during the interval $(x, x+dx)$ given that the elapsed vacation time is $x$.

$\gamma_j(x) = \frac{h_j(x)}{1-H_j(x)}; j = 1(ER), 2(OR)$ that the conditional probability completion of repair period during the interval $(x, x+dx)$ given that the elapsed repair time is $x$.

At time $t$, let $M(t)$ be the number of customers in the waiting line and $\xi(t)$ be the supplementary variable at time $t$. The $\xi(t)$ have the following random identifications. $\xi = \xi_i$; i = 0, during elapsed ES period; i = 1, during elapsed OS period; i = 2, during elapsed vacation period; i = 3, during elapsed ER period i = 4, during elapsed OR period.

The two-dimensional process $\{(M(t),\xi(t)):t\geq 0\}$ is a Markov process. The following probabilities and probability generating functions are introduced for the analysis:

$$Q_n(t)=\Pr\{M(t)=n,\text{the server is idle}\},n=0,1,\ldots,K-1.$$

$$P_{n1}(x;t)=\Pr\{M(t)=n,\xi_0(t)\in(x,x+\Delta t)\},n=0,1,\ldots.$$

$$P_{n2}(x;t)=\Pr\{M(t)=n,\xi_1(t)\in(x,x+\Delta t)\},n=0,1,\ldots.$$

$$V_n(x;t)=\Pr\{M(t)=n,\xi_2(t)\in(x,x+\Delta t)\},n=0,1,\ldots.$$

$$R_{n1}(x;t)=\Pr\{M(t)=n,\xi_3(t)\in(x,x+\Delta t)\},n=0,1,\ldots.$$

$$R_{n2}(x;t)=\Pr\{M(t)=n,\xi_4(t)\in(x,x+\Delta t)\},n=0,1,\ldots.$$

In steady state,

$$P_{ni}(x)=\lim_{n\to\infty}P_{ni}(x;t)\,;i=1,2\;;\;\;V_n(x)=\lim_{n\to\infty}V_n(x;t),\;R_{ni}(x)=\lim_{n\to\infty}R_{ni}(x;t),$$

$$C(z)=\sum_{j=1}^{\infty}C_jz^j\,,P_i(x,z)=\sum_{n=0}^{\infty}P_{ni}(x)z^n\,;i=1,2\,,V(x,z)=\sum_{n=0}^{\infty}V_n(x)z^n,$$

$$R_{ni}(x,z)=\sum_{n=0}^{\infty}R_{ni}(x)z^n\,,Q(z)=\sum_{n=0}^{K-1}Q_nz^n;\text{ where }|z|\leq 1$$

## 3.The Analysis

The system discussed $M^{[X]}/G^K/1$, the following differential-difference equations are obtained using the supplementary variable technique as outlined in Cox (1965).

$$\frac{dP_{01}(x)}{dx}=-(\lambda_1+\mu_1(x)+\alpha)P_{01}(x) \tag{1a}$$

$$\frac{dP_{n1}(x)}{dx}=-(\lambda_1+\mu_1(x)+\alpha)P_{n1}(x)+\lambda_1\sum_{j=1}^{n}C_jP_{n-j1}(x)\,,n\geq 1 \tag{1b}$$

$$\frac{dP_{02}(x)}{dx}=-(\lambda_2+\mu_2(x)+\alpha)P_{02}(x) \tag{2a}$$

$$\frac{dP_{n2}(x)}{dx}=-(\lambda_2+\mu_2(x)+\alpha)P_{n2}(x)+\lambda_2\sum_{j=1}^{n}C_jP_{n-j2}(x)\,,n\geq 1 \tag{2b}$$

$$\frac{dV_0(x)}{dx}=-(\lambda_3+\beta(x))V_0(x) \tag{3a}$$

$$\frac{dV_n(x)}{dx}=-(\lambda_3+\beta(x))V_n(x)+\lambda_3\sum_{j=1}^{n}C_jV_{n-j}(x)\,,n\geq 1 \tag{3b}$$

$$\frac{dR_{01}(x)}{dx}=-(\lambda_4+\gamma_1(x))R_{01}(x) \tag{4a}$$

$$\frac{dR_{n1}(x)}{dx}=-(\lambda_4+\gamma_1(x))R_{n1}(x)+\lambda_4\sum_{j=1}^{n}C_jR_{n-j1}(x)\,,n\geq 1 \tag{4b}$$

$$\frac{dR_{02}(x)}{dx}=-(\lambda_5+\gamma_2(x))R_{02}(x) \tag{5a}$$

$$\frac{dR_{n2}(x)}{dx} = -\big(\lambda_5 + \gamma_2(x)\big)R_{n2}(x) + \lambda_5 \sum_{j=1}^{n} C_j R_{n-j2}(x) \,, n \geq 1 \qquad (5b)$$

$$0 = -\lambda_0 Q_n + \lambda_0\big(1 - \delta_{n,0}\big)\sum_{j=1}^{n} C_j Q_{n-j} + (1-r)\int_0^{\infty} \gamma_i(x)\, R_{ni}(x)dx$$
$$+ \int_0^{\infty} \beta(x)V_n(x)dx; \;\; n = 0,1,\dots,K-1 \;, i = 1,2 \qquad (6)$$

The boundary conditions are,

$$P_{n1}(0) = \int_0^{\infty} \beta(x)V_{n+K}(x)dx + (1-r)\int_0^{\infty} \gamma_i(x)R_{n+K}(x)dx$$
$$+\lambda_0 \sum_{j=0}^{K-1} C_{n+K-j}Q_j \,; \;\; n \geq 0 \qquad (7a)$$

$$P_{n2}(0) = \int_0^{\infty} \mu_1(x)P_{n+K1}(x)dx \qquad (7b)$$

$$V_n(0) = (1-q)\int_0^{\infty} \mu_1(x)P_{n+K1}(x)dx + \int_0^{\infty} \mu_2(x)P_{n+K2}(x)dx; \qquad n \geq 0 \qquad (8)$$

$$R_{n1}(0) = \alpha(1-r)\left[(1-q)\int_0^{\infty} P_{n-K1}(x)dx + \int_0^{\infty} P_{n+K2}(x)dx\right]; \; n \geq K \qquad (9a)$$

$$R_{n2}(0) = \alpha r\left[(1-q)\int_0^{\infty} P_{n-K1}(x)dx + \int_0^{\infty} P_{n+K2}(x)dx\right]; \; n \geq K \qquad (9b)$$

$$R_{n1}(0) = R_{n2}(0) = 0; n < K \qquad (9c)$$

and the normalization condition is

$$\sum_{n=0}^{K-1} Q_n + \int_0^{\infty} \sum_{n=0}^{\infty} [P_{n1}(x) + P_{n2}(x) + V_n(x) + R_{n1}(x) + R_{n2}(x)] \; dx = 1 \qquad (10)$$

**Theorem 3.1:**

Under steady state condition, the model has the following probability generating functions.

$P_1(z) = \frac{mQ(z)a_1 z^K u_1}{J}; P_2(z) = \frac{mQ(z)a_1 G_1^*(a_1)u_2}{J}; V(z) = \frac{mQ(z)a_1 a_2 G_1^*(a_1)u_3}{Jm_1}$

$R_1(z) = \frac{\alpha z^K mQ(z)u_4 u_7(1-r)}{Jm_2}; R_2(z) = \frac{rz^K mQ(z)u_5 u_7}{Jm_3}$

where,

$u_1 = [1 - G_1^*(a_1)], u_2 = [1 - G_2^*(a_2)], u_3 = [1 - B^*(m_1)], u_4 = [1 - H_1^*(m_1)],$
$u_5 = \big[1 - H_2^*(m_2)\big], u_6 = \{z^K(1-q) + G_2^*(a_2)\}, u_7 = u_1 z^K(1-q)a_2 + a_1 G_1^*(a_1)u_2$
$m = \lambda_0 - \lambda_0 C(z), J = \alpha z^K(1-r)H_1^*(m_2)u_7 + rz^K H_2^*(m_3)u_7 - a_1 a_2[z^{2K} - B^*(m_1)$
$G_1^*(a_1)u_6], a_1 = \lambda_1 - \lambda_1 C(z) + \alpha, a_2 = \lambda_2 - \lambda_2 C(z) + \alpha, m_1 = \lambda_3 - \lambda_3 C(z)m_2 = \lambda_4 -$
$\lambda_4 C(z), m_3 = \lambda_5 - \lambda_5 C(z), O = a_1 a_2[z^{2K} - B^*(m_1)G_1^*(a_1), u_6 - \alpha z^K(1-r)\, H_1^*(m_2)$

$u_7 - rz^K H_2^*(m_3)u_7], e_1 = [(1-q)\, P_1(0,z)\big[1 - G_1^*(a_1)\big] + \, P_2(0,z)\, \big[1 - G_2^*(a_2)\big]]$
respectively, the probability generating function of number of customers in queue when the server provides ES, when the server provides OS, when the server provides is on vacation, when the server provides is in ER and when the server provides is in OR.

***Proof:***

Multiplication of equations (1a) and (1b) by appropriate powers of $z$ and adding the resultant equations for $n = 0,1,\ldots\infty$ , leads to

$$\frac{\partial}{\partial x}\big(\, P_1(x,z)\big) + (\lambda_1 - \lambda_1 C(z) + \mu_1(x) + \alpha)\, P_1(x,z) = 0 \tag{11}$$

Multiplication of equations (2a) and (2b) by appropriate powers of $z$ and adding the resultant equations for $n = 0,1,\ldots\infty$ , leads to

$$\frac{\partial}{\partial x}\big(\, P_2(x,z)\big) + (\lambda_2 - \lambda_2 C(z) + \mu_2(x) + \alpha)\, P_2(x,z) = 0 \tag{12}$$

Multiplication of equations (3a) and (3b) by appropriate powers of $z$ and adding the resultant equations for $n = 0,1,\ldots\infty$ , leads to

$$\frac{\partial}{\partial x}\big(V(x,z)\big) + \big(\lambda_3 - \lambda_3 C(z) + \beta(x)\big)V(x,z) = 0 \tag{13}$$

Multiplication of equations (4a) and (4b) by appropriate powers of $z$ and adding the resultant equations for $n = 0,1,\ldots\infty$ , leads to

$$\frac{\partial}{\partial x}\big(R_1(x,z)\big) + \big(\lambda_4 - \lambda_4 C(z) + \gamma_1(x)\big)R_1(x,z) = 0 \tag{14}$$

Multiplication of equations (5a) and (5b) by appropriate powers of $z$ and adding the resultant equations for $n = 0,1,\ldots\infty$ , leads to

$$\frac{\partial}{\partial x}\big(R_2(x,z)\big) + \big(\lambda_5 - \lambda_5 C(z) + \gamma_2(x)\big)R_2(x,z) = 0 \tag{15}$$

Multiplication of equation (7a) by $z^{n+K}$ and summation over $n = 0,1,\ldots\infty$ , leads to

$$\begin{aligned} z^K\, P_1(0,z) = &\int_0^\infty \beta(x) \sum_{n=K}^{\infty} V_n(x) z^n\, dx + K(z) + (1-r) \\ &+ \left[\int_0^\infty \gamma_1(x) \sum_{n=K}^{\infty} R_{n1}(x) z^n\, dx + \int_0^\infty \gamma_2(x) \sum_{n=K}^{\infty} R_{n2}(x) z^n\, dx \right] \end{aligned} \tag{16}$$

where, $K(z) = \sum_{j=0}^{K-1} \sum_{n=0}^{\infty} z^{n+K}\, C_{n+K-j} Q_j$

Multiplication of equation (6) by $z^n$ and summation over $n = 0,1, \ldots K-1$ , leads to

$$0 = -\lambda_0 Q(z) + L(z) + \int_0^\infty \beta(x) \sum_{n=0}^{K-1} V_n(x) z^n dx + (1-r)$$
$$+ \left[ \int_0^\infty \gamma_1(x) \sum_{n=K}^{\infty} R_{n1}(x) z^n\, dx + \int_0^\infty \gamma_2(x) \sum_{n=K}^{\infty} R_{n2}(x) z^n\, dx \right] \quad (17)$$

where, $L(z) = \lambda_0 \left(1 - \delta_{n,0}\right) \sum_{j=1}^{n} \sum_{n=0}^{K-1} z^n\, C_j Q_{n-j}$

Multiplication of equation (7b) by $z^{n+K}$ and summation over $n = 0,1, \ldots \infty$ , leads to

$$P_2(0,z) = \frac{\int_0^\infty P_1(x,z)\mu_1(x)dx}{z^K} \quad (18)$$

Now, addition of equations (16) and (17), we have

$$z^K\, P_1(0,z) = \int_0^\infty V(x,z)\beta(x)dx + \lambda_0[C(z)-1]Q(z)$$
$$+ (1-r)\left[\int_0^\infty R_1(x,z)\gamma_1(x)dx + \int_0^\infty R_2(x,z)\gamma_2(x)dx \right] \quad (19)$$

Multiplication of equation (8) by $z^n$ and summation over $n = 0,1, \ldots \infty$ , leads to

$$V(0,z) = (1-q)\int_0^\infty P_1(x,z)\mu_1(x)dx + \int_0^\infty P_2(x,z)\mu_2(x)dx \quad (20)$$

Multiplication of equations (9a) and (9c) by appropriate powers of $z$ and adding the resultant equations for $n = 0,1, \ldots \infty$ , leads to

$$R_1(0,z) = (1-r)\alpha\, z^K[(1-q)P_1(z) + P_2(z)] \quad (21)$$

Multiplication of equations (9b) and (9c) by appropriate powers of $z$ and adding the resultant equations for $n = 0,1, \ldots \infty$ , leads to

$$R_2(0,z) = r\, z^K[(1-q)P_1(z) + P_2(z)] \quad (22)$$

Integrating of equation (11) leads to,

$$P_1(x,z) = P_1(0,z)e^{-a_1 x - \int_0^\infty \mu_1(x)dx} \quad (23)$$

Integrating of equation (23) leads to,

$$\int_0^\infty P_1(x,z)dx = P_1(z) = \frac{P_1(0,z)[1 - G_1^*(a_1)]}{a_1} \quad (24)$$

Multiplying of equation (23) by $\mu_1(x)$ and integration of the equation leads to,

$$\int_0^\infty P_1(x,z)\mu_1(x)dx = P_1(0,z)G_1^*(a_1) \quad (25)$$

Integrating of equation (12) leads to,

$$P_2(x,z) = P_2(0,z)e^{-a_2 x - \int_0^\infty \mu_2(x)dx} \quad (26)$$

Integrating of equation (26) leads to,

$$\int_0^\infty P_2(x,z)dx = P_2(z) = \frac{P_2(0,z)[1 - G_2^*(a_2)]}{a_2} \tag{27}$$

Multiplying of equation (26) by $\mu_2(x)$ and integration of the equation leads to,

$$\int_0^\infty P_2(x,z)\mu_2(x)dx = P_2(0,z)G_2^*(a_2) \tag{28}$$

Substituting the value of equation (25), (28) in (20), we have

$$V(0,z) = (1-q)\, P_1(0,z)G_1^*(a_1) + P_2(0,z)G_2^*(a_2) \tag{29}$$

Integrating of equation (13) leads to,

$$V(x,z) = V(0,z)e^{-m_1x - \int_0^\infty \beta(x)dx} \tag{30}$$

Substituting the value of equation (29) in (30), we have

$$V(x,z) = e^{-m_1x - \int_0^\infty \beta(x)dx}\{\, (1-q)\, P_1(0,z)G_1^*(a_1) + P_2(0,z)G_2^*(a_2)\, \} \tag{31}$$

Integrating of equation (31) leads to,

$$\int_0^\infty V(x,z)dx = V(z) = \frac{\{\, (1-q)\, P_1(0,z)G_1^*(a_1) + P_2(0,z)G_2^*(a_2)\, \}[1 - B^*(m_1)]}{m_1} \tag{32}$$

Multiplying of equation (31) by $\beta(x)$ and integration of the equation leads to,

$$\int_0^\infty V(x,z)\beta(x)dx = \{\, (1-q)\, P_1(0,z)G_1^*(a_1) + P_2(0,z)G_2^*(a_2)\, \}B^*(m_1) \tag{33}$$

Integrating of equation (14) leads to,

$$R_1(x,z) = R_1(0,z)e^{-m_2x - \int_0^\infty \gamma_1(x)dx} \tag{34}$$

Substituting the value of equation (24), (27), (21) in (34), we have

$$R_1(x,z) = (1-r)\alpha\, z^K e_1\, e^{-m_2x - \int_0^\infty \gamma_1(x)dx} \tag{35}$$

Integrating of equation (35) leads to,

$$\int_0^\infty R_1(x,z)dx = R_1(z) = \frac{(1-r)\alpha\, z^K e_1[1 - H_1^*(m_2)]}{a_1a_2m_2} \tag{36}$$

Multiplying of equation (35) by $\gamma_1(x)$ and integration of the equation leads to,

$$\int_0^\infty R_1(x,z)\gamma_1(x)dx = \frac{(1-r)\alpha\, z^K H_1^*(m_2)e_1}{a_1a_2} \tag{37}$$

Integrating of equation (15) leads to,

$$R_2(x,z) = R_2(0,z)e^{-m_3x - \int_0^\infty \gamma_2(x)dx} \tag{38}$$

Substituting the value of equation (24), (27), (22) in (38), we have

$$R_2(x,z) = rz^K e_1\, e^{-m_3 x - \int_0^\infty \gamma_2(x)dx} \quad (39)$$

Integrating of equation (39) leads to,

$$\int_0^\infty R_2(x,z)dx = R_2(z) = \frac{r\, z^K e_1[1 - H_1^*(m_3)]}{a_1 a_2 m_3} \quad (40)$$

Multiplying of equation (39) by $\gamma_2(x)$ and integration of the equation leads to,

$$\int_0^\infty R_2(x,z)\gamma_2(x)dx = \frac{r\, z^K H_1^*(m_3)e_1}{a_1 a_2} \quad (41)$$

Substituting the value of equation (25) in (18), we have

$$P_2(0,z) = \frac{P_1(0,z)G_1^*(a_1)}{z^K} \quad (42)$$

Substituting the value of equations (33), (37), (41), (42) in (19), we have

$$P_1(0,z) = \frac{mQ(z)a_1 a_2 z^K}{J} \quad (43)$$

Substituting the value of equation (43) in (42), we have

$$P_2(0,z) = \frac{mQ(z)a_1 a_2 G_1^*(a_1)}{J} \quad (44)$$

Substituting the value of equation (43) in (24), we have

$$P_1(z) = \frac{mQ(z)a_1 z^K u_1}{J} \quad (45)$$

Substituting the value of equation (44) in (27), we have

$$P_2(z) = \frac{mQ(z)a_1 G_1^*(a_1)u_2}{J} \quad (46)$$

Substituting the value of equations (43) and (44) in (32), we have

$$V(z) = \frac{mQ(z)a_1 a_2 G_1^*(a_1)u_3}{Jm_1} \quad (47)$$

Substituting the value of equations (43) and (44) in (36), we have

$$R_1(z) = \frac{\alpha z^K mQ(z)u_4 u_7(1-r)}{Jm_2} \quad (48)$$

Substituting the value of equations (43) and (44) in (40), we have

$$R_2(z) = \frac{rz^K mQ(z)u_5 u_7}{Jm_3} \quad (49)$$

**Theorem 3.2:**

Under the steady state condition, the probability generating function for number of customers in the queue is $S(z)$, where $S(z) = Q(z) + N(z)$.

***Proof:***

Let $S(z) = Q(z) + P_1(z) + P_2(z) + V(z) + R_1(z) + R_2(z)$ (50)

$N(z) = P_1(z) + P_2(z) + V(z) + R_1(z) + R_2(z)$

Now, Substituting the value of equation (45), (46), (47), (48), and (49) in (50), we have

$$S(z) = \frac{Q(z)X}{Jm_1m_2m_3} \quad (51)$$

where, $X = m\{m_1m_2m_3\{a_1z^Ku_1 + a_1G_1^*(a_1)u_2\} + m_2m_3a_1a_2G_1^*(a_1)u_3 + m_1m_3\alpha z^Ku_4u_7(1-r) + m_2m_1rz^Ku_5u_7\} + Jm_1m_2m_3$

$S(z) = \frac{A}{z-z_0}$ by substituting $z = 1$, we get

$A = (1-z_0)S(1)$

$$S(1) = \frac{Q(f_1+f_2)}{f_2} \quad (52)$$

where, $a_3 = \lambda_0E(X), a_4 = \lambda_1E(X), a_5 = \lambda_2E(X), a_6 = \lambda_3E(X), a_7 = \lambda_4E(X)$

$a_8 = \lambda_5E(X), u_8 = [1 - G_1^*(\alpha)], u_9 = [1 - G_2^*(\alpha)], u_{11} = G_1^*(\alpha)[(1-q) + G_2^*(\alpha)]$

$u_{10} = u_8(1-q) + G_1^*(\alpha)u_9, u_{12} = [K - a_6E(V)], u_{13} = [1 - G_2^*(\alpha)G_1^*(\alpha)]$

$f_1 = a_3\{u_1 + u_2G_1^*(\alpha)\} + \alpha(1-r)a_3u_{10}E(R_1) + ra_3u_{10}E(R_2) + E(V)a_3G_1^*(\alpha)u_{11}$

$f_2 = \alpha G_1^*(\alpha)u_{11}u_{12} - a_4(1-q)u_1 - a_5u_{13} - \alpha(1-r)a_7u_{10}E(R_1) - ra_8u_{10}E(R_2)$

By applying Rouche's theorem,

Substituting the value of $S(1)$ in the above equation, we have

$$A = \frac{(1-z_0)Q(f_1+f_2)}{f_2} \quad (53)$$

Substituting the value of (53) in the above equation, we have

$$S(z) = \frac{(z_0-1)Q(f_1+f_2)}{z_0f_2}\sum_{n=0}^{\infty}\left(\frac{z}{z_0}\right)^n \quad (54)$$

which is the probability generating function of number of customers in the queue.

### 3.1 Some system measures

To shows the performance of the model, the following system measures are derived:

1. The idle probability is $Q = \sum_{n=0}^{K-1} Q_n$ which leads to, $Q = \frac{(f_2-f_1)}{f_2}$

   This is obtained by using $Q + N(1) = 1 \Rightarrow Q = 1 - N(1)$
2. The average number of customers in the queue when the server provides ES

$$N_1 = P_1'(1) = \frac{Q\{4K\alpha a_3u_8 - \lambda_0u_{14}\alpha u_8 - 4a_3a_5u_8 - 4a_3u_{15}\}}{4f_2}$$

where, $u_{14} = (E(X)^2 - E(X)), u_{15} = a_4 G_1^{*'}(\alpha), u_{16} = a_5 G_2^{*'}(\alpha)$

3. The average number of customers in the queue when the server provides OS

$$N_2 = P_2'(1) = \frac{Q\{4a_3u_{15}u_9 - \lambda_0 u_{14}u_9 G_1^*(\alpha) + 4a_3u_{16}G_1^*(\alpha) - 4a_3a_4u_9G_1^*(\alpha)\}}{f_2}$$

4. The average number of customers in the queue when the server is on compulsory vacation

$$N_3 = V(1) = \frac{Qa_3f_3}{24a_6f_2}$$

where, $u_{17} = [(1-q) + G_2^*(\alpha)], u_{18} = [(1-q)K + G_2^*(\alpha)], f_3 = a_3u_{11}[E(V)\lambda_3u_{14} +(E(V)E(X)\lambda_3)^2] + \lambda_0\alpha^2u_{14}u_{11}a_6E(V) + 4a_3\alpha u_{11}a_6E(V)[a_5 + a_4] + 4a_3a_4u_{17}a_6\alpha^2 E(S_1)E(V) - 4a_3\alpha^2u_{18}a_6\, E(V)G_1^*(\alpha) + 4a_3\alpha^2G_1^*(\alpha)[(1-q)K + E(S_2)a_5]E(V)a_6 -3a_3\alpha^2u_{11}E(V)\lambda_3E(X)^2$

5. The average number of customers in the queue when the server is on essential repair

$$N_4 = R_1'(1) = \frac{Qf_4(1-r)}{24a_7f_2}$$

where, $f_4 = 4\alpha a_3a_7u_{10}\, E(R_1) + 4\alpha^2a_3a_7u_{10}\, E(R_1)\{(1-q)a_4\, E(S_1) + u_9G_1^*(\alpha)\} +4a_3a_7E(R_1)\{u_8(1-q) + a_4u_9\, E(S_1)\} + 4a_3a_7E(R_1)\{u_8(1-q) + a_5G_1^*(\alpha)\, E(S_1)\} -\alpha^2\lambda_0u_{14}u_{10}a_7 - a_3\alpha^2u_{10}[E(R_1)\lambda_4u_{14} + (E(R_1)E(X)\lambda_4)^2] - 4Ka_3\alpha^2u_{10}a_7E(R_1) -4a_3a_7\{Ku_8(1-q) + u_9G_1^*(\alpha)\}\alpha^2 - 3\alpha^2a_3a_7u_{10}\, E(R_1)E(X)^2\lambda_4$

6. The average number of customers in the queue when the server is on optional repair

$$N_5 = R_2'(1) = \frac{Qf_5r}{24a_8f_2}$$

where, $f_5 = \alpha\lambda_0u_{14}a_8u_{10}\, E(R_2) + \alpha u_{10}a_3a_8[E(R_2)\lambda_5u_{14} + (E(R_2)E(X)\lambda_5)^2] -4K\alpha u_{10}a_3a_8\, E(R_2) - 4a_3a_8\, E(R_2)\{a_5u_8(1-q) + \alpha u_9G_1^*(\alpha)\} - 4a_3a_8\, E(R_2) \{\alpha u_8(1-q) + u_9u_4G_1^*(\alpha)\} - 4a_3a_8\, E(R_2)\{(1-q)E(S_1)a_4 + u_9G_1^*(\alpha)\} - 4a_3 a_8\, E(R_2)\{u_8(1-q) + a_4u_9\, E(S_1)\} - 4u_{10}a_3a_8E(R_2) - 3u_{10}a_3a_8E(R_2)E(X)^2\lambda_5$

7. The average number of customers in the queue is $S = S'(1) = \frac{Q(f_1+f_2)}{(1-z_0)f_2}$
8. The server's utilization factor is $\rho = 1 - Q$
9. Mean waiting time of a customer is $W = \frac{S}{\lambda^*}$
   where, $\lambda^*$ is the effective arrival rate and

$$\lambda^* = Q\lambda_0 + P_1(1)\lambda_1 + P_2(1)\lambda_2 + V(1)\lambda_3 + R_1(1)\lambda_4 + R_2(1)\lambda_5$$

## 4. Numerical illustrations

In this section, numerical study for the model discussed in this paper is carried out by assuming service times, vacation times and repair time as negative exponential distribution and batch size as geometric distribution, The parameter values are $C_j = \delta(1-\delta)^{j-1}, j = 1,2,\dots; 0 < \delta < 1;\; E(X) = \frac{(1-\delta)}{\delta}, E(V) = \frac{1}{\beta}, E(S_i) = \frac{1}{\mu_i}, E(R_i) = \frac{1}{\gamma_i}, G_i^*(\alpha) = \frac{\mu_i}{\alpha+\mu_i}, i = 1,2$

$$B^*(m_1) = \frac{\beta}{\beta + m_1}, H_1^*(m_2) = \frac{\gamma_1}{\gamma_1 + m_2}, H_2^*(m_3) = \frac{\gamma_2}{\gamma_2 + m_3}$$

**TABLE 1: System Measures $\alpha = 0.6, \beta = 1.0, \delta = 0.47, q = 0.4, z = 0.89,$ $K = 16, \lambda_1 = 1.7, \lambda_2 = 2.0, \lambda_3 = 1.4, \lambda_4 = 1.9, \lambda_5 = 1.5, \mu_1 = 0.7, \mu_2 = 0.3,$ $\gamma_1 = 1.1, \gamma_1 = 1.0$**

| $\lambda_0$ | $\rho$ | $Q$ | $N_1$ | $N_2$ | $N_3$ | $N_4$ | $N_5$ | $S$ |
|---|---|---|---|---|---|---|---|---|
| 1.1 | 0.6969 | 0.3656 | 4.5082 | 2.3464 | 0.0325 | 0.4028 | 0.6762 | 1.4893 |
| 1.2 | 0.7201 | 0.3860 | 5.1925 | 2.7026 | 0.0374 | 0.4630 | 0.7788 | 1.4106 |
| 1.3 | 0.7432 | 0.4051 | 5.9041 | 3.0730 | 0.0425 | 0.5256 | 0.8855 | 1.3439 |
| 1.4 | 0.7664 | 0.4231 | 6.6404 | 3.4562 | 0.0478 | 0.5904 | 0.9960 | 1.2868 |
| 1.5 | 0.7896 | 0.4400 | 7.3993 | 3.8512 | 0.0533 | 0.6570 | 1.1098 | 1.2373 |
| 1.6 | 0.8128 | 0.4560 | 8.1788 | 4.2570 | 0.0589 | 0.7255 | 1.2267 | 1.1940 |
| 1.7 | 0.8360 | 0.4711 | 8.9773 | 4.6726 | 0.0646 | 0.7955 | 1.3465 | 1.1558 |
| 1.8 | 0.8591 | 0.4853 | 9.7931 | 5.0972 | 0.0705 | 0.8671 | 1.4689 | 1.1219 |
| 1.9 | 0.8823 | 0.4988 | 10.6250 | 5.5302 | 0.0765 | 0.9401 | 1.5936 | 1.0915 |
| 2.0 | 0.9055 | 0.5117 | 11.4717 | 5.9709 | 0.0826 | 1.0143 | 1.7206 | 1.0641 |

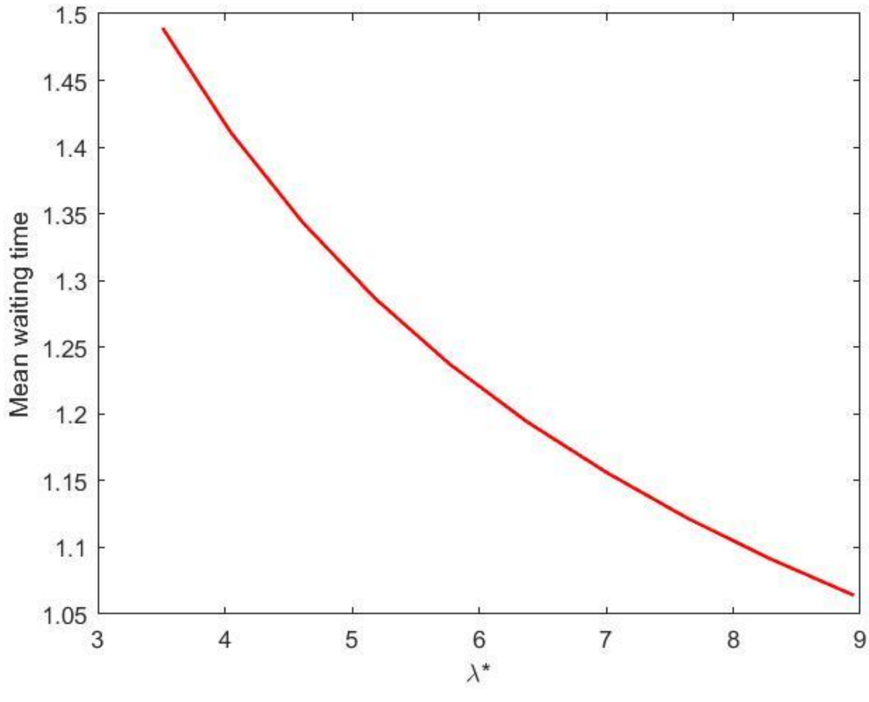


**Figure 1: Mean waiting time against $\lambda^*$**

Incorporating the above expression in the formulas in 3.1. The numerical values of the measures, the ideal probability, the average number of customers, the average number of customers in the queue when the server provides ES, OS, compulsory vacation, ER, OR, the server's utilization factor and the mean waiting time of a customer are obtained using MATLAB and the values are presented in table 1 and figure 1. This numerical illustration is to show the practical applicability of the analytical results derived in this paper.

## 5. Conclusion

This study corresponds to a single unreliable server non-Markovian queue. The customers arrive in batches of variable size follows compound Poisson process and the services are given batches of fixed size follows a general distribution. The arrival rates are state dependent. The server applies compulsory vacation policy; vacation period is generally distributed. Breakdown occurs when the server is busy, the number of breakdowns follows Poisson process. Immediately the server undergoes two types of repairs called essential and optional. Both repair periods are generally distributed. The system contains a waiting line of infinite capacity. The model is completely analyzed in steady state. To show the versatile of the model some numerical illustrations are provided. The model can be generalized by assuming the general arrival process.

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