$L(3, 1)$ Labeling of Some Graph Families of Line Graph of Crown Graph

K. K. Raval[1], H. A. Pandey[*]
and N. B. Patel[2]

[1] Department of Mathematics, Gujarat University, Ahmedabad, Gujarat
[*]Science & Humanities Department, Silver Oak College of Engineering & Technology, Silver Oak University, Ahmedabad, Gujarat
[2] Science & Humanities Department, SAL Institute of Technology & Engineering Research, Ahmedabad, Gujarat

Abstract

An $L(3, 1)$ - labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all positive integers such that $|f(x) - f(y)| \geq 3$ if $d(x, y) = 1$ akd $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$ where for all $x, y \in V(G)$. The $L(3, 1)$ labeling number of graph G, denoted by $\lambda(G)$ or λ, is the smallest positive integer m such that $max. \{f(x) : x \in V(G)\} = m$. This paper totally involves the calculation of λ of line graph of crown graphs, duplication of all the vertices of degree two of line graph of a crown graph by an edge, duplication of all the vertices of degree two of line graph of crown graph by a vertex and a graph obtained by connecting two copies of line graph of crown graph by a path.

Keywords: Crown graph, duplication, line graph, $L(3, 1)$ labeling, $\lambda(G)$.
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1. Introduction

An $L(3, 1)$ labeling of graphs is basically motivated from the channel assignment problem which was introduced by Hale [6].It is also used in solving the frequency assignment problem in most of the wireless network stations. Different frequencies are assigned in such a way so that the transmitter does not interfere with one another. This frequency assignment problem is similar to labeling an $L(3, 1)$ label to each vertex. Throughout we use a simple, finite, undirected graph G, with vertex set $V(G)$ and edge set $E(G)$. For different graph labeling techniques, we use a dynamic survey of graph labeling by Gallian[1]. For various notation and terminology we follow Gross and Yellen [5].

Definition 1. 1. For graph G with w, $v \in V(G)$, and for fixed positive integers j akd k where $k \leq$ j, the function $L : V(G) \rightarrow Z^+$ is called $L(j, k) -$ labeling of G if and only if $|L(v) - L(w)| \geq$ j if v akd w are adjacent and $|L(v) - L(w)| \geq k$ if v akd w are distance two apart. It was first introduced by Griggs and Yeh [4].

Definition 1. 2. Let G be a graph with set of vertices $V(G)$ and set of edges $E(G)$. Let f be a function $f : V \rightarrow Z^+$, where f is $L(3,1) -$ labeling [3] of G if, for all $u, v \in V(G)$, $|f(u) - f(v)| \geq 3$ if $d(u, v)$ 1 akd $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$.

Definition 1. 3. The line graph $L(G)$ [7] of a graph G has vertices expressive edges of G, and two vertices are nearby in $L(G)$ if and only if analogous edges are nearby in G.

Definition 1. 4. The crown graph Cr_n [1] is obtained by joining pendant vertices $\{v'_i, i = 0, 1, 2, \ldots, k - 1\}$ with each consecutive vertex $\{u'_i,$ i = 0, 1, 2, . . . , n – 1$\}$ of the cycle graph C_n. The edge set of crown graph is $\{u'_i u'_{i+1} = u_i, i = 0, 1, 2, \ldots, k - 2\} \cup \{u'_{n-1}u_n = u_{n-1}\} \cup \{u'_i v'_i = v_i, i = 0, 1, 2, \ldots, k - 1\}$.

Definition 1. 5. The line graph of crown graph $L(Cr_n)$ is constructed from Cr_n. A cycle $u_0, u_1, u_2, \ldots, u_{n-1}, u_0$ is formed using the edges of Cr_n. Each vertex v_i is adjacent to u_i and u_{i+1}, where i is considered modulo $k - 1$. Thus, $|V(L(Cr_n))| = 2k$ akd $|E(L(Cr_n))| = 3k$. Throughout the article, we use same labels for further investigations.

Definition 1. 6. Duplication of u vertices by a new edge e $= u'u''$ in a graph G generates a new graph G' such that $N_G(u') = \{u, u''\}$ akd $N_{G^F}(u'') = \{u, u'\}$.

Definition 1.7. Duplication of vertex u by a new vertex v forms a new graph G' such that $N_G(u) = N_{G'}(v)$, where $N_G(u)$ is the set of all the vertices adjacent to u in graph G.

2. Results

Theorem 2.1. Line graph of crown graph is an $L(3,1)$ labeled graph and $\lambda(L(Cr_n)) = 10$ for $k \geq 3$.
Proof. Define a function $f : V(G) \to Z^+$, where $G = L(Cr_n)$, let the consecutive edges and vertices of crown graph Cr_n be e_i and u_i where $0 \leq i \leq k-1$ respectively of C_n. Let u_i and u_{i+1} be joined with a vertex v_i to attain the $L(Cr_n)$. Thus, $V(L(Cr_n)) = \{u_i, v_i : i = 0, 1, 2, \ldots, k-1\}$. Thus, following subsequent cases arise on applying $L(3,1)$ labeling on $L(Cr_n)$:

Case 1: $n \equiv 0 \pmod 4$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, \quad i = 4k+1, \quad k = 0,1,2,\ldots\ldots,\frac{k-4}{4}; \\ 3, & \text{if } x = u_i, \quad i = 4k+2, \quad k = 0,1,2,\ldots\ldots,\frac{k-4}{4}; \\ 6, & \text{if } x = u_i, \quad i = 4k+3, \quad k = 0,1,2,\ldots\ldots,\frac{k-4}{4}; \\ 9, & \text{if } x = u_i, \quad i = 4k, \quad k = 1,2,\ldots\ldots,\frac{k}{4}; \\ 7, & \text{if } x = v_i, \quad i = 4k+1, \quad k = 0,1,2,\ldots\ldots,\frac{k-4}{4}; \\ 10, & \text{if } x = v_i, \quad i = 4k+2, \quad k = 0,1,2,\ldots\ldots,\frac{k-4}{4}; \\ 1, & \text{if } x = v_i, \quad i = 4k+3, \quad k = 0,1,2,\ldots\ldots,\frac{k-4}{4}; \\ 5, & \text{if } x = v_i, \quad i = 4k, \quad k = 1,2,\ldots\ldots,\frac{k}{4}. \end{cases}$$

Case 2: $n \equiv 1 \pmod 4$.
Subcase 2.1: $n = 5$.

$$f(x) = \begin{cases} 3(i-1), & \text{if } x = u_i, \quad i = 1,2,3,4; \\ 4, & \text{if } x = u_5; \\ 8, & \text{if } x = v_1; \\ 10, & \text{if } x = v_2; \\ 2, & \text{if } x = v_3; \\ 1, & \text{if } x = v_4; \\ 7, & \text{if } x = v_5. \end{cases}$$

Subcase 2.2: $k > 5$

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, \quad i = 4k+1, \quad k = 0,1,2,\ldots\ldots,\frac{k-5}{4}; \\ 3, & \text{if } x = u_i, \quad i = 4k+2, \quad k = 0,1,2,\ldots\ldots,\frac{k-5}{4}; \\ 6, & \text{if } x = u_i, \quad i = 4k+3, \quad k = 0,1,2,\ldots\ldots,\frac{k-5}{4}; \\ 9, & \text{if } x = u_i, \quad i = 4k, \quad k = 1,2,\ldots\ldots,\frac{k}{4}; \\ 4, & \text{if } x = u_n; \\ 8, & \text{if } x = v_i, \quad i = 4k+1, \quad k = 0,1,2,\ldots\ldots,\frac{k-5}{4}; \\ 10, & \text{if } x = v_i, \quad i = 4k+2, \quad k = 0,1,2,\ldots\ldots,\frac{k-5}{4}; \\ 1, & \text{if } x = v_i, \quad i = 4k+3, \quad k = 0,1,2,\ldots\ldots,\frac{k-5}{4}; \\ 5, & \text{if } x = v_i, \quad i = 4k, \quad k = 1,2,\ldots\ldots,\frac{k}{4}; \\ 7, & \text{if } x = v_n. \end{cases}$$

Case 3: n ≡ 2 (mod 4).

Subcase 3.1: n ≡ 6 (mod 12).

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots,\frac{k-3}{3}; \\ 3, & \text{if } x = u_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots,\frac{k-3}{3}; \\ 6, & \text{if } x = u_i, \quad i = 3k, \quad k = 1,2,\dots\dots,\frac{k}{3}; \\ 7, & \text{if } x = v_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots,\frac{k-3}{3}; \\ 9, & \text{if } x = v_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots,\frac{k-3}{3}; \\ 10, & \text{if } x = v_i, \quad i = 3k, \quad k = 1,2,\dots\dots,\frac{k}{3}. \end{cases}$$

Subcase 3.2: $k \equiv 10 \ (mod\ 12)$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots,\frac{k-4}{3}; \\ 3, & \text{if } x = u_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots,\frac{k-4}{3}; \\ 6, & \text{if } x = u_i, \quad i = 3k, \quad k = 1,2,\dots\dots,\frac{k}{3}; \\ 9, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots,\frac{k-4}{3}; \\ 9, & \text{if } x = v_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots,\frac{k-4}{3}; \\ 10, & \text{if } x = v_i, \quad i = 3k, \quad k = 1,2,\dots\dots,\frac{k-4}{3}; \\ 10, & \text{if } x = v_{n-2}; \\ 1, & \text{if } x = v_{n-1}; \\ 4, & \text{if } x = v_n. \end{cases}$$

Subcase 3.3: $k \equiv 2 \ (mod\ 12)$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots,\frac{k-5}{3}; \\ 3, & \text{if } x = u_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots,\frac{k-5}{3}; \\ 6, & \text{if } x = u_i, \quad i = 3k, \quad k = 1,2,\dots\dots,\frac{k-2}{3}; \\ 9, & \text{if } x = u_{n-1}; \\ 4, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots,\frac{k-5}{3}; \\ 9, & \text{if } x = v_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots,\frac{k-8}{3}; \\ 10, & \text{if } x = v_i, \quad i = 3k, \quad k = 1,2,\dots\dots,\frac{k-5}{3}; \\ 10, & \text{if } x = v_{n-3}; \\ 2, & \text{if } x = v_{n-2}; \\ 1, & \text{if } x = v_{n-1}; \\ 8, & \text{if } x = v_n. \end{cases}$$

Case 4: n ≡ 3 (mod 4).

Subcase 4.1: $k = 3$.

$$f(x) = \begin{cases} 3(i-1), & \text{if } x = u_i, \quad i = 1,2,3; \\ 7, & \text{if } x = v_1; \\ 9, & \text{if } x = v_2; \\ 10, & \text{if } x = v_3; \end{cases}$$

Subcase 4.2: $k \equiv 7 \ (mod\ 12)$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots, \frac{k-4}{3}; \\ 3, & \text{if } x = u_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots, \frac{k-4}{3}; \\ 6, & \text{if } x = u_i, \quad i = 3k, \quad k = 1,2,\dots\dots, \frac{k-1}{3}; \\ 9, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots, \frac{k-4}{3}; \\ 9, & \text{if } x = v_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots, \frac{k-7}{3}; \\ 10, & \text{if } x = v_i, \quad i = 3k, \quad k = 1,2,\dots\dots, \frac{k-4}{3}; \\ 10, & \text{if } x = v_{n-2}; \\ 1, & \text{if } x = v_{n-1}; \\ 4, & \text{if } x = v_n. \end{cases}$$

Subcase 4.3: $k \equiv 11 \ (mod\ 12)$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots, \frac{k-5}{3}; \\ 3, & \text{if } x = u_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots, \frac{k-5}{3}; \\ 6, & \text{if } x = u_i, \quad i = 3k, \quad k = 1,2,\dots\dots, \frac{k-2}{3}; \\ 9, & \text{if } x = u_{n-1}; \\ 4, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots, \frac{k-5}{3}; \\ 9, & \text{if } x = v_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots, \frac{k-8}{3}; \\ 10, & \text{if } x = v_i, \quad i = 3k, \quad k = 1,2,\dots\dots, \frac{k-5}{3}; \\ 10, & \text{if } x = v_{n-3}; \\ 2, & \text{if } x = v_{n-2}; \\ 1, & \text{if } x = v_{n-1}; \\ 8, & \text{if } x = v_n. \end{cases}$$

Subcase 4.4: $n \equiv 3 \ (mod\ 12)$ and $n > 3$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots, \frac{k-3}{3}; \\ 3, & \text{if } x = u_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots, \frac{k-3}{3}; \\ 6, & \text{if } x = u_i, \quad i = 3k, \quad k = 1,2,\dots\dots, \frac{k}{3}; \\ 7, & \text{if } x = v_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots, \frac{k-3}{3}; \\ 9, & \text{if } x = v_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots, \frac{k-3}{3}; \\ 10, & \text{if } x = v_i, \quad i = 3k, \quad k = 1,2,\dots\dots, \frac{k}{3}. \end{cases}$$

Thus, by all the above cases it is clear that $L(3,1)$ labeling is satisfied for $L(Cr_n)$ and $\lambda\ (L(Cr_n))$ for each case is 10.

Illustration 2.2. $L(3,1)$ labeling of $L(Cr_5)$ is shown in the figure below for $\lambda = 10$.

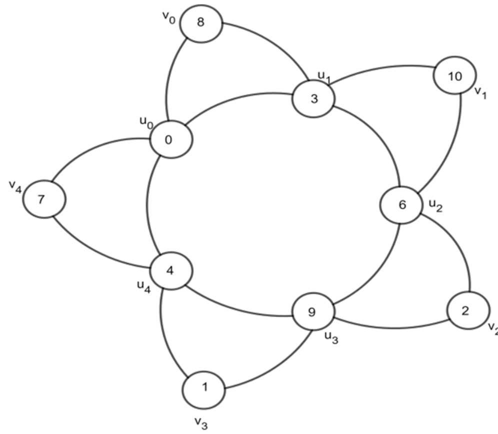

Figure 1: $L(3,1)$ labeling of $L(Cr_5)$

Theorem 2. 3**.** The graph G obtained by duplication of all the vertices of degree two of $L(Cr_n)$ by an edge admits $L(3,1)$ labeling for $k \geq 3$ and $\lambda(G) = 10$.

Proof. The graph G obtained by joining all the outer vertices v_i, $i = 0, 1, 2, \dots, n-1$ of $L(Cr_n)$ by an edge having the end vertices w_i and w_{i+1}, where i is considered as modulo $k - 1$. Thus $|V(G)| = 4k$ $|E(G)| = 6k$. Define a function $f : V(G) \rightarrow Z^+$ such that:

Case 1: $n \equiv 0 \pmod 3$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots,\frac{k-3}{3}; \\ 3, & \text{if } x = u_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots,\frac{k-3}{3}; \\ 6, & \text{if } x = u_i, \quad i = 3k, \quad k = 1,2,\dots\dots,\frac{k}{3}; \\ 7, & \text{if } x = v_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots,\frac{k-3}{3}; \\ 9, & \text{if } x = v_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots,\frac{k-3}{3}; \\ 10, & \text{if } x = v_i, \quad i = 3k, \quad k = 1,2,\dots\dots,\frac{k}{3}; \\ 1, & \text{if } x = w_i, \quad i = 2k+1, \quad k = 0,1,2,\dots\dots,k-1; \\ 4, & \text{if } x = w_i, \quad i = 2k, \quad k = 1,2,\dots\dots,k. \end{cases}$$

Case 2: $k \equiv 1 \pmod 3$.

Subcase 2.1: $k = 4$.

$$f(x) = \begin{cases} 3(i-1), & \text{if } x \in \{u_1, u_2, u_3, u_4\}; \\ 7, & \text{if } x = v_1; \\ 10, & \text{if } x = v_2; \\ 1, & \text{if } x \in \{v_3, w_1, w_3, w_8\}; \\ 4, & \text{if } x \in \{v_4, w_2, w_4, w_6\}; \\ 8, & \text{if } x \in \{w_{2n-3}, w_{2n-1}\}. \end{cases}$$

Subcase 2.2: $k > 4$.

$$f(x)=\begin{cases} 0, & \text{if } x = u_i, \quad i = 3k+1, \quad k = 0,1,2,\ldots\ldots,\frac{k-4}{3};\\ 3, & \text{if } x = u_i, \quad i = 3k+2, \quad k = 0,1,2,\ldots\ldots,\frac{k-4}{3};\\ 6, & \text{if } x = u_i, \quad i = 3k, \quad k = 1,2,\ldots\ldots,\frac{k-1}{3};\\ 9, & \text{if } x = u_n;\\ 7, & \text{if } x = v_i, \quad i = 3k+1, \quad k = 0,1,2,\ldots\ldots,\frac{k-4}{3};\\ 10, & \text{if } x = v_i, \quad i = 3k+2, \quad k = 0,1,2,\ldots\ldots,\frac{k-4}{3};\\ 9, & \text{if } x = v_i, \quad i = 3k, \quad k = 1,2,\ldots\ldots,\frac{k-4}{3};\\ 1, & \text{if } x \in \{v_{n-1}, w_{2n-1}\};\\ 4, & \text{if } x \in \{v_n, w_{2n-3}\};\\ 1, & \text{if } x = w_i, \quad i = 2k+1, \quad k = 0,1,2,\ldots\ldots,k-3;\\ 4, & \text{if } x = w_i, \quad i = 2k, \quad k = 1,2,\ldots\ldots,k-2;\\ 8, & \text{if } x \in \{w_{2n-2}, w_{2n}\}. \end{cases}$$

Case 3: n $\equiv$ 2 (mod 3).

Subcase 3.1: n $=$ 5.

$$f(x)=\begin{cases} 3(i-1), & \text{if } x \in \{u_1,u_2,u_3,u_4\};\\ 4, & \text{if } x \in \{u_5, w_2, w_4\};\\ 7, & \text{if } x = v_1;\\ 10, & \text{if } x = v_2;\\ 2, & \text{if } x = v_3;\\ 1, & \text{if } x \in \{v_4, w_1, w_3, w_{2n}\};\\ 5, & \text{if } x \in \{w_{2n-5}, w_{2n-3}, w_{2n-1}\};\\ 8, & \text{if } x = w_i, \quad i \in \{v_5, w_{2n-4}, w_{2n-2}\}. \end{cases}$$

Subcase 3.2: n $>$ 5.

$$f(x)=\begin{cases} 0, & \text{if } x = u_i, \quad i = 3k+1, \quad k = 0,1,2,\ldots\ldots,\frac{k-5}{3};\\ 3, & \text{if } x = u_i, \quad i = 3k+2, \quad k = 0,1,2,\ldots\ldots,\frac{k-5}{3};\\ 6, & \text{if } x = u_i, \quad i = 3k, \quad k = 1,2,\ldots\ldots,\frac{k-2}{3};\\ 9, & \text{if } x = u_{n-1};\\ 4, & \text{if } x = u_n;\\ 7, & \text{if } x = v_i, \quad i = 3k+1, \quad k = 0,1,2,\ldots\ldots,\frac{k-5}{3};\\ 10, & \text{if } x = v_i, \quad i = 3k+2, \quad k = 0,1,2,\ldots\ldots,\frac{k-5}{3};\\ 9, & \text{if } x = v_i, \quad i = 3k, \quad k = 1,2,\ldots\ldots,\frac{k-5}{3};\\ 2, & \text{if } x = v_{n-2};\\ 1, & \text{if } x \in \{v_{n-1}, w_{2n}\};\\ 1, & \text{if } x = w_i, \quad i = 2k+1, \quad k = 0,1,2,\ldots\ldots,k-4;\\ 4, & \text{if } x = w_i, \quad i = 2k, \quad k = 1,2,\ldots\ldots,k-3;\\ 5, & \text{if } x \in \{w_{2n-5}, w_{2n-3}, w_{2n-1}\};\\ 8, & \text{if } x = w_i, \quad i \in \{v_5, w_{2n-4}, w_{2n-2}\}. \end{cases}$$

Thus, by above cases it is clear that $L(3,1)$ labeling is satisfied for duplication of outer vertices of $L(Cr_n)$ by an edge and λ number for each case is 10.

Theorem 2.4**.** The graph G obtained by duplication of all the vertices of degree two of $L(Cr_n)$ by a vertex admits $L(3,1)$ labeling with $\lambda(G) = 12$ for $k \equiv 0\ (mod\ 6)$ and $\lambda(G) = 13$ otherwise.

Proof. Let G be a graph obtained by duplicating each of the vertices of degree two $\{v_i, i = 0, 1, 2, \ldots, k-1\}$ by a new vertex $\{w_i, i = 0, 1, 2, \ldots, k-1\}$, where w_i are adjacent to w_i and u_{i+1} for $i = 0, 1, 2, \ldots, k-1$. Thus, $|V(G)| = 3k, |E(G)| = 5k$. Define a function $f : V(G) \to Z^+$ such that:

Case 1: $k \equiv 0 \ (mod\ 3)$.

Subcase 1.1: $k = 3$.

$$f(x) = \begin{cases} 3(i-1), & \text{if } x = u_i, \quad i = 1,2,3; \\ 7, & \text{if } x = v_1; \\ 9, & \text{if } x = v_2; \\ 10, & \text{if } x = v_3; \\ 11, & \text{if } x = w_1; \\ 12, & \text{if } x = w_2; \\ 13, & \text{if } x = w_3. \end{cases}$$

Subcase 1.2: $k \equiv 0 \ (mod\ 6)$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, \quad i = 3k+1, \quad k = 0,1,2,\ldots\ldots, \frac{k-3}{3}; \\ 3, & \text{if } x = u_i, \quad i = 3k+2, \quad k = 0,1,2,\ldots\ldots, \frac{k-3}{3}; \\ 6, & \text{if } x = u_i, \quad i = 3k, \quad k = 1,2,\ldots\ldots, \frac{k}{3}; \\ 7, & \text{if } x = v_i, \quad i = 3k+1, \quad k = 0,1,2,\ldots\ldots, \frac{k-3}{3}; \\ 9, & \text{if } x = v_i, \quad i = 3k+2, \quad k = 0,1,2,\ldots\ldots, \frac{k-3}{3}; \\ 10, & \text{if } x = v_i, \quad i = 3k, \quad k = 1,2,\ldots\ldots, \frac{k}{3}; \\ 11, & \text{if } x = w_i, \quad i = 2k+1, \quad k = 0,1,2,\ldots\ldots, \frac{k-2}{2}; \\ 12, & \text{if } x = w_i, \quad i = 2k, \quad k = 1,2,\ldots\ldots, \frac{k}{2} \end{cases}$$

Subcase 1.3: $k \equiv 3 \ (mod\ 6)$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, \quad i = 3k+1, \quad k = 0,1,2,\ldots\ldots, \frac{k-3}{3}; \\ 3, & \text{if } x = u_i, \quad i = 3k+2, \quad k = 0,1,2,\ldots\ldots, \frac{k-3}{3}; \\ 6, & \text{if } x = u_i, \quad i = 3k, \quad k = 1,2,\ldots\ldots, \frac{k}{3}; \\ 7, & \text{if } x = v_i, \quad i = 3k+1, \quad k = 0,1,2,\ldots\ldots, \frac{k-3}{3}; \\ 9, & \text{if } x = v_i, \quad i = 3k+2, \quad k = 0,1,2,\ldots\ldots, \frac{k-3}{3}; \\ 10, & \text{if } x = v_i, \quad i = 3k, \quad k = 1,2,\ldots\ldots, \frac{k}{3}; \\ 11, & \text{if } x = w_i, \quad i = 2k+1, \quad k = 0,1,2,\ldots\ldots, \frac{k-3}{2}; \\ 12, & \text{if } x = w_i, \quad i = 2k, \quad k = 1,2,\ldots\ldots, \frac{k-1}{2}; \\ 13, & \text{if } x = w_n. \end{cases}$$

Case 2: $k \equiv 1 \ (mod\ 3)$.

Subcase 2.1: $k = 4$.

$$f(x) = \begin{cases} 3(\mathrm{i}-1), & \text{if } x \in \{u_1, u_2, u_3, u_4\}; \\ 7, & \text{if } x = v_1; \\ 10, & \text{if } x = v_2; \\ 1, & \text{if } x = v_3; \\ 4, & \text{if } x = v_4; \\ 11, & \text{if } x = w_1; \\ 12, & \text{if } x \in \{w_2, w_4\}; \\ 13, & \text{if } x = w_3. \end{cases}$$

Subcase 2.2: $k \equiv 1 \ (mod\ 6)$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots,\frac{k-4}{3}; \\ 3, & \text{if } x = u_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots,\frac{k-4}{3}; \\ 6, & \text{if } x = u_i, \quad i = 3k, \quad k = 1,2,\dots\dots,\frac{k-1}{3}; \\ 9, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots,\frac{k-4}{3}; \\ 10, & \text{if } x = v_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots,\frac{k-4}{3}; \\ 9, & \text{if } x = v_i, \quad i = 3k, \quad k = 1,2,\dots\dots,\frac{k-4}{3}; \\ 1, & \text{if } x = v_{n-1}; \\ 4, & \text{if } x = v_n; \\ 11, & \text{if } x = w_i, \quad i = 2k+1, \quad k = 0,1,2,\dots\dots,\frac{k-3}{2}; \\ 12, & \text{if } x = w_i, \quad i = 2k, \quad k = 1,2,\dots\dots,\frac{k-1}{2}; \\ 13, & \text{if } x = w_n. \end{cases}$$

Subcase 2.3: $k \equiv 4 \ (mod\ 6)$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots,\frac{k-4}{3}; \\ 3, & \text{if } x = u_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots,\frac{k-4}{3}; \\ 6, & \text{if } x = u_i, \quad i = 3k, \quad k = 1,2,\dots\dots,\frac{k-1}{3}; \\ 9, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots,\frac{k-4}{3}; \\ 10, & \text{if } x = v_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots,\frac{k-4}{3}; \\ 9, & \text{if } x = v_i, \quad i = 3k, \quad k = 1,2,\dots\dots,\frac{k-4}{3}; \\ 1, & \text{if } x = v_{n-1}; \\ 4, & \text{if } x = v_n; \\ 11, & \text{if } x = w_i, \quad i = 2k+1, \quad k = 0,1,2,\dots\dots,\frac{k-4}{2}; \\ 12, & \text{if } x = w_i, \quad i = 2k, \quad k = 1,2,\dots\dots,\frac{k}{2}; \\ 13, & \text{if } x = w_{n-1}. \end{cases}$$

Case 3: $k \equiv 2 \ (mod\ 3)$.

Subcase 3.1: $k \equiv 5 \ (mod\ 6)$.

Subsubcase 3.1.1: $k = 5$.

$$f(x) = \begin{cases} 3(i-1), & \text{if } x \in \{u_1, u_2, u_3, u_4\}; \\ 4, & \text{if } x = u_n; \\ 7, & \text{if } x = v_1; \\ 10, & \text{if } x = v_2; \\ 2, & \text{if } x = v_3; \\ 1, & \text{if } x = v_4; \\ 8, & \text{if } x = v_5; \\ 11, & \text{if } x = w_1; \\ 12, & \text{if } x \in \{w_2, w_4\}; \\ 13, & \text{if } x \in \{w_3, w_5\}. \end{cases}$$

Subsubcase 3.1.2: $k > 5$.

$$f(x) = \begin{cases} 0, & \text{if } x = u_i, \quad i = 3k+1, \quad k = 0, 1, 2, \dots \dots, \frac{k-5}{3}; \\ 3, & \text{if } x = u_i, \quad i = 3k+2, \quad k = 0, 1, 2, \dots \dots, \frac{k-5}{3}; \\ 6, & \text{if } x = u_i, \quad i = 3k, \quad k = 1, 2, \dots \dots, \frac{k-2}{3}; \\ 9, & \text{if } x = u_{n-1}; \\ 4, & \text{if } x = u_n; \\ 7, & \text{if } x = v_i, \quad i = 3k+1, \quad k = 0, 1, 2, \dots \dots, \frac{k-5}{3}; \\ 10, & \text{if } x = v_i, \quad i = 3k+2, \quad k = 0, 1, 2, \dots \dots, \frac{k-5}{3}; \\ 9, & \text{if } x = v_i, \quad i = 3k, \quad k = 1, 2, \dots \dots, \frac{k-5}{3}; \\ 2, & \text{if } x = v_{n-2}; \\ 1, & \text{if } x = v_{n-1}; \\ 8, & \text{if } x = v_n; \\ 11, & \text{if } x = w_i, \quad i = 2k+1, \quad k = 0, 1, 2, \dots \dots, \frac{k-3}{2}; \\ 12, & \text{if } x = w_i, \quad i = 2k, \quad k = 1, 2, \dots \dots, \frac{k-1}{2}; \\ 13, & \text{if } x \in \{w_{n-1}, w_n\}. \end{cases}$$

Subcase 3.2: $k \equiv 2 \ (mod\ 6)$

$$f(x)=\begin{cases} 0, & \text{if } x = u_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots,\frac{k-5}{3};\\ 3, & \text{if } x = u_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots,\frac{k-5}{3};\\ 6, & \text{if } x = u_i, \quad i = 3k, \quad k = 1,2,\dots\dots,\frac{k-2}{3};\\ 9, & \text{if } x = u_{n-1};\\ 4, & \text{if } x = u_n;\\ 7, & \text{if } x = v_i, \quad i = 3k+1, \quad k = 0,1,2,\dots\dots,\frac{k-5}{3};\\ 10, & \text{if } x = v_i, \quad i = 3k+2, \quad k = 0,1,2,\dots\dots,\frac{k-5}{3};\\ 9, & \text{if } x = v_i, \quad i = 3k, \quad k = 1,2,\dots\dots,\frac{k-5}{3};\\ 2, & \text{if } x = v_{n-2};\\ 1, & \text{if } x = v_{n-1};\\ 8, & \text{if } x = v_n;\\ 11, & \text{if } x = w_i, \quad i = 2k+1, \quad k = 0,1,2,\dots\dots,\frac{k-4}{2};\\ 12, & \text{if } x = w_i, \quad i = 2k, \quad k = 1,2,\dots\dots,\frac{k}{2};\\ 13, & \text{if } x = w_{n-1}. \end{cases}$$

Thus, by above all the cases it is clear that the graph G obtained by duplication of inner vertices of degree two of $L(Cr_n)$ by a vertex admits $L(3,1)$ labeling and $\lambda(G) = 12$ for $k \equiv 0 \ (mod\ 6)$ and $\lambda(G) = 13$ otherwise.

Theorem 2.5**.** The graph G' obtained by connecting two copies of $L(Cr_n)$ for all $k \geq 3$, by a path P_m, where $m \geq 4$ is $L(3,1)$ graph and $\lambda = 11$.
Proof. For $L(Cr_n)$ $|V(L(Cr_n))| = 2k$ and $|E(L(Cr_n))| = 3k$ and for the graph G'obtained by connecting two copies of $L(Cr_n)$ by a path P_m, where $m \geq 4$ and first and last vertices of P_m, are connected to u_0 of each copy of $L(Cr_n)$. Name vertices of P_m as $p_1, p_2, \dots\dots, p_{m-2}$ other than first and last vertices of P_m. Thus, $|V(G')| = 4k + m - 2$ and $|E(G')| = 6k + m - 1$. We refer to the labeling of $L(Cr_n)$ as in theorem 3.1 with the same function $f: V(G) \to Z^+$. Now define a new function $f': V(G') \to Z^+$such that:

Case 1: $m \equiv 0 \ (mod\ 3)$.

$$f'(x)=\begin{cases} f(x), & \text{if } x \in V(L(Cr_n));\\ 5, & \text{if } x = p_{3k+1}, \quad k = 0,1,2,\dots\dots,\frac{m-3}{3};\\ 8, & \text{if } x = p_{3k+2}, \quad k = 0,1,2,\dots\dots,\frac{m-6}{3};\\ 11, & \text{if } x = p_{3k}, \quad k = 1,2,\dots\dots,\frac{m-3}{3}. \end{cases}$$

Case 2: $m \equiv 1 \ (mod\ 3)$.

Subcase 2.1: $m = 4$.

$$f'(x)=\begin{cases} f(x), & \text{if } x \in V(L(Cr_n));\\ 5, & \text{if } x = p_1;\\ 11, & \text{if } x = p_2. \end{cases}$$

Subcase 2.2: $m > 4$.

$$f(x) = \begin{cases} f(x), & \text{if } x \in V(L(Cr_n)); \\ 5, & \text{if } x = p_{3k+1}, \quad k = 0, 1, 2, \dots \dots, \frac{m-4}{3}; \\ 8, & \text{if } x = p_{3k+2}, \quad k = 0, 1, 2, \dots \dots, \frac{m-4}{3}; \\ 11, & \text{if } x = p_{3k}, \quad k = 1, 2, \dots \dots, \frac{m-4}{3}. \end{cases}$$

Case 3: $m \equiv 2 \ (mod\ 3)$.

$$f(x) = \begin{cases} f(x), & \text{if } x \in V(L(Cr_n)); \\ 5, & \text{if } x = p_{3k+1}, \quad k = 0, 1, 2, \dots \dots, \frac{m-5}{3}; \\ 8, & \text{if } x = p_{3k+2}, \quad k = 0, 1, 2, \dots \dots, \frac{m-5}{3}; \\ 11, & \text{if } x = p_{3k}, \quad k = 1, 2, \dots \dots, \frac{m-2}{3}. \end{cases}$$

Thus, by above all cases it is clear that the graph G' obtained by connecting two copies of $L(Cr_n)$ for all $k \geq 3$, by a path P_m, where $m \geq 4$, admits $L(3, 1)$ labeling and $\lambda = 11$.

3. Conclusion

Line graph of crown graph is constructed and $L(3, 1)$ had been applied on it. $L(3, 1)$ labeling on $L(Cr_n)$ results in $\lambda = 10$. All the vertices of degree two are duplicated by an edge and the obtained graph admits $L(3, 1)$ labeling for $\lambda = 10$. Also $\lambda = 12$ for $k \equiv 0 \ (mod\ 6)$ and $\lambda = 13$ otherwise for the graph obtained by duplicating each of its vertices of degree two by a new vertex. New graph is constructed by joining two copies of $L(Cr_n)$ by a path P_m which admits $L(3, 1)$ labeling and $\lambda = 11$. $L(3, 1)$ labeling can be applied on more graph families which can be constructed using crown graph and different graph operations like corona product, complement of a graph, shadow graph and many more.

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